THE CAMBRIDGE

MATHEMATICAL JOURNAL.

Vol. I.] NOVEMBER, 1837. [No. I.

I.—NOTES ON THE UNDULATORY THEORY OF LIGHT.

[No. I.]

The following demonstrations of some Theorems in Fresnel's Memoir on Double Refraction, are not offered as part of a complete treatise on this part of the Undulatory Theory, but merely as shewing that some of the remarkable propositions deduced by that author may be conveniently proved by a shorter and more direct analysis than that which he has employed. Fresnel generally makes use of a mixed Geometry, which was perhaps the best method for establishing his theorems at first; but as his proofs are often tedious, it seems desirable to obtain demonstrations more suited to the general style of mathematics in researches of this kind.

1. We shall begin with a demonstration of the existence of a system of three axes of elasticity—a proposition on which the whole theory of double refraction depends, and which Fresnel has proved by a method which has the advantage of geometrical distinctness, but which is long and rather difficult to follow out on that account. The proposition is thus stated:—*In any system of particles acting on each other with forces which are functions of their mutual distances, there are three directions at right angles to each other, along which if a particle be displaced, the forces of restitution will act in the same direction.*

Let x, y, z be the coordinates of the attracted point,

$x_1, y_1, z_1, x_2, y_2, z_2, \ldots$ be the coordinates of the attracting points,

$r_1, r_2, r_3, \ldots$ the distances between the attracted and attracting points,

$\phi_1(r_1), \phi_2(r_2), \phi_3(r_3), \ldots$ the attractions,

X, Y, Z, the total resolved forces along the axes;

then we shall have

$$X = \frac{x_1 - x}{r_1}\,\phi_1(r_1) + \frac{x_2 - x}{r_2}\,\phi_2(r_2) + \dots$$

And similarly for Y and Z. Now let

$$R = -\,\Sigma\!\int\!\phi(r)\,dr.$$

$$\left.\begin{aligned} \text{Then}\quad X &= \frac{dR}{dx} = 0,\\ Y &= \frac{dR}{dy} = 0,\\ Z &= \frac{dR}{dz} = 0. \end{aligned}\right\}\ \text{when the particle is in equilibrio,}$$

Let the particle receive a small displacement, the projections of which on the coordinate axes are δx, δy, δz. Then supposing the displacement to be very small, the force of restitution may be taken as proportional to it, so that we have

$$X = \frac{d^2R}{dx^2}\delta x + \frac{d^2R}{dx\,dy}\delta y + \frac{d^2R}{dx\,dz}\delta z,$$

$$Y = \frac{d^2R}{dy\,dx}\delta x + \frac{d^2R}{dy^2}\delta y + \frac{d^2R}{dy\,dz}\delta z,$$

$$Z = \frac{d^2R}{dz\,dx}\delta x + \frac{d^2R}{dz\,dy}\delta y + \frac{d^2R}{dz^2}\delta z.$$

Now the force of restitution will be in the direction of the displacement, if X, Y, Z, be proportional to δx, δy, δz. Let then

$$s = \frac{X}{\delta x} = \frac{Y}{\delta y} = \frac{Z}{\delta z}.$$

Then putting

$$\frac{d^2R}{dx^2} = A, \qquad \frac{d^2R}{dy^2} = B, \qquad \frac{d^2R}{dz^2} = C,$$

$$\frac{d^2R}{dz\,dy} = \frac{d^2R}{dy\,dz} = A', \quad \frac{d^2R}{dz\,dx} = \frac{d^2R}{dx\,dz} = B', \quad \frac{d^2R}{dx\,dy} = \frac{d^2R}{dy\,dx} = C';$$

and substituting in the former equations, they become

$$(A-s)\delta x + C'\delta y + B'\delta z = 0,$$
$$C'\delta x + (B-s)\delta y + A'\delta z = 0,$$
$$B'\delta x + A'\delta y + (C-s)\delta z = 0.$$

Eliminating δx, δy, δz, by cross multiplication,* we obtain, as an equation of condition,

$$(A-s)(B-s)(C-s) - A'^2(A-s) - B'^2(B-s) - C'^2(C-s) + 2A'B'C' = 0.$$

This is obviously the same equation as that found in investigating the existence of three principal diametral planes in surfaces of the

* For an explanation of this method, see the last Article in this Number.

second order, as well as of the three principal axes of rotation. As it is a cubic equation, it has at least one real root. Let us suppose that the axis of z is that which corresponds to this root, so that a displacement along it produces a force of restitution acting in the same direction. In this case A′ and B′ will vanish, for A′ is the force along the axis of y arising from a displacement along z, and B′ is the corresponding quantity for x. The equations are thus reduced to

$$(\mathrm{A}-s) + \mathrm{C}' \frac{\delta y}{\delta x} = 0,$$

$$(\mathrm{B}-s) + \mathrm{C}' = 0.$$

Eliminating s, we get

$$\left(\frac{\delta y}{\delta x}\right)^2 + \frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}'}\frac{\delta y}{\delta x} - 1 = 0.$$

The last term of this quadratic being -1, the two lines whose directions are determined by the two values of $\frac{\delta y}{\delta x}$ are at right angles to each other, and as $\delta z = 0$, they are in the plane of xy; consequently there are three directions at right angles to each other, along which, if a particle be displaced, the force of restitution acts in the same direction.

2. The next proposition we shall prove, is that for determining the velocities of the waves of light in a crystal. But it will be necessary first to recal Fresnel's construction for finding them.

Having proved, as we have just done, that in every crystal there are three axes of elasticity passing through every point, it is a natural supposition, confirmed by experiment, that these axes are always parallel to fixed straight lines. Take therefore these axes as the coordinate axes, and let the forces excited by displacements equal to unity in these directions be a^2, b^2, c^2, respectively. Then if a particle receive a displacement $= 1$ in a direction making angles X, Y, Z, with these axes, the resolved parts of the displacement will be

$$\cos \mathrm{X},\ \cos \mathrm{Y},\ \cos \mathrm{Z},$$

and the resolved parts of the force will be

$$a^2 \cos \mathrm{X},\ b^2 \cos \mathrm{Y},\ c^2 \cos \mathrm{Z};$$

so that if f be the whole force,

$$f = \sqrt{a^4 \cos^2 \mathrm{X} + b^4 \cos^2 \mathrm{Y} + c^4 \cos^2 \mathrm{Z}},$$

and the cosines of the angles which its direction makes with the axes are

$$\frac{a^2 \cos \mathrm{X}}{f},\ \frac{b^2 \cos \mathrm{Y}}{f},\ \frac{c^2 \cos \mathrm{Z}}{f},$$

and the cosine of the angle between the direction of displacement and the direction of the force of restitution will be

$$\frac{a^2 \cos^2 \mathrm{X} + b^2 \cos^2 \mathrm{Y} + c^2 \cos^2 \mathrm{Z}}{f}.$$

And if the force be resolved along, and perpendicular to, the direction of the displacement, the former part will be

$$a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z.$$

If now we construct a surface whose equation is

$$r^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z,$$

and a particle be displaced along any radius, the square of that radius will represent the resolved part, in that direction, of the force of restitution. This surface is called the surface of elasticity, and its equation between rectangular coordinates evidently is

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = r^4 = (x^2 + y^2 + z^2)^2.$$

When the part of the force which is perpendicular to the direction of displacement is also perpendicular to the front of the wave, Fresnel has shewn that it will produce no effect, and therefore may be neglected. But this will not generally be the case, and therefore this force will be equivalent to two others—one perpendicular to the front of the wave, which may be neglected, and another in the plane of the front of the wave, which will have to be compounded with that in the direction of the displacement; so that, in general, the total effective force will not be in the direction of the displacement. It will be shewn, however, that in every plane there are two directions at right angles to each other, in which, if a particle be displaced, the part of the force perpendicular to the line of displacement will also be perpendicular to the plane. If a displacement take place in any other direction in that plane, we may resolve it in those two directions, so that the forces excited will be in the same directions and proportional to the squares of the corresponding radii of the surface of elasticity, and vibrations parallel to these directions will traverse the medium with velocities proportional to the radii.

To prove the existence of these two directions, let

$$lx + my + nz = 0$$

be the equation to the plane of the front of the wave,

$$l'x + m'y + n'z = 0$$

the equation to a plane passing through the directions of the displacement and of the excited force, so that

$$l' \cos X + m' \cos Y + n' \cos Z = 0 \quad \ldots\ldots\ldots(1),$$

$$\text{and } l' a^2 \cos X + m' b^2 \cos Y + n' c^2 \cos Z = 0 \quad \ldots\ldots\ldots(2).$$

In order that the part of the force which is not in the direction of the displacement may be perpendicular to the front of the wave, these two planes must be perpendicular to each other; therefore

$$ll' + mm' + nn' = 0 \quad \ldots\ldots\ldots(3).$$

Eliminating l', m', n', between these three equations, by cross multiplication, we get

$$(b^2 - c^2)\, l \cos Y \cos Z + (c^2 - a^2)\, m \cos Z \cos X$$
$$+ (a^2 - b^2)\, n \cos X \cos Y = 0 \quad \ldots\ldots\ldots(4);$$

which, together with the equations

$$l \cos X + m \cos Y + n \cos Z = 0 \;\ldots\ldots\ldots(5)$$

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1 \;\ldots\ldots\ldots(6),$$

determine the angles X, Y, Z, and therefore the direction of the displacement.

These directions are the same as those of the greatest and least radii of a section of the surface of elasticity made by the same plane. For to determine these we have the equation

$$r^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z,$$

with the condition $dr = 0$, and the equations (5) and (6).

Differentiating, we get

$$a^2 \cos X\, d \cos X + b^2 \cos Y\, d \cos Y + c^2 \cos Z\, d \cos Z = 0,$$

$$l\, d \cos X + m\, d \cos Y + n\, d \cos Z = 0,$$

$$\cos X\, d \cos X + \cos Y\, d \cos Y + \cos Z\, d \cos Z = 0.$$

Eliminating, as before, $d \cos X$, $d \cos Y$, $d \cos Z$, between these three equations, we get

$$(b^2 - c^2)\, l \cos Y \cos Z + (c^2 - a^2)\, m \cos Z \cos X + (a^2 - b^2)\, n \cos X \cos Y = 0;$$

the same as (4), so that X, Y, Z, being determined by the same equations in the two cases, the resulting values will be the same in both. Hence, in order to find the velocities of the rays of light in passing through a crystal, we have merely to determine the greatest and least radii of a section of the surface of elasticity. This may be effected more readily than is done by Fresnel, in the following manner.

Let $$r^4 = a^2 x^2 + b^2 y^2 + c^2 z^2 \;\ldots\ldots\ldots(1)$$

be the equation to the surface of elasticity, where

$$r^2 = x^2 + y^2 + z^2 \;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and let the surface be cut by a plane

$$0 = lx + my + nz \;\ldots\ldots\ldots\ldots\ldots(3).$$

When r is the greatest or least radius in the section made by this plane, we have the condition $dr = 0$. Differentiating the three equations with this condition, we get

$$0 = a^2 x\, dx + b^2 y\, dy + c^2 z\, dz \;\ldots\ldots\ldots(4)$$

$$0 = x\, dx + y\, dy + z\, dz \;\ldots\ldots\ldots(5)$$

$$0 = l\, dx + m\, dy + n\, dz \;\ldots\ldots\ldots(6).$$

Then A (6) + B (5) + (4) gives, on equating to 0 the coefficients of each differential

$$Al + Bx + a^2 x = 0$$

$$Am + By + b^2 y = 0$$

$$An + Bz + c^2 z = 0.$$

Multiply by x, y, z, and add, considering the conditions (1), (2), (3). Then we have

$$Br^2 + r^4 = 0, \quad \text{or } B = -r^2;$$

therefore substituting

$$\mathrm{A}l = (r^2-a^2)\,x, \quad \mathrm{A}m = (r^2-b^2)\,y, \quad \mathrm{A}n = (r^2-c^2)\,z,$$

$$\text{or } x = \frac{\mathrm{A}l}{r^2-a^2}, \qquad y = \frac{\mathrm{A}m}{r^2-b^2}, \qquad z = \frac{\mathrm{A}n}{r^2-c^2}.$$

Multiply by l, m, n, and add; then, by the condition (3) we have

$$\frac{l^2}{r^2-a^2} + \frac{m^2}{r^2-b^2} + \frac{n^2}{r^2-c^2} = 0,$$

a quadratic equation in r^2 from which two values of r^2 may be found, and thus the velocities determined.

It is easy from this to determine the equation to the wave surface, for it is the locus of the ultimate intersections of planes the perpendiculars on which from the origin are determined by the above equation. Calling the perpendicular v, it will therefore be determined by the following equations:

$$lx + my + nz = v \;\ldots\ldots\ldots(1),$$

$$\text{and} \quad \frac{l^2}{v^2-a^2} + \frac{m^2}{v^2-b^2} + \frac{n^2}{v^2-c^2} = v \;\ldots\ldots\ldots(2),$$

$$\text{where} \quad l^2 + m^2 + n^2 = 1 \;\ldots\ldots\ldots(3).$$

For the process employed the reader is referred to the *Cambridge Transactions*, vol. vi. part I. But we may add here a method of finally eliminating l, m, n, and v, which is somewhat shorter than that employed there.

Having found that

$$x\,(a^2-v^2) = lv\,(a^2-r^2), \quad y\,(b^2-v^2) = mv\,(b^2-r^2),$$
$$z\,(c^2-v^2) = nv\,(c^2-r^2),$$

substitute the values of l, m, n, given by these equations in (2), which then becomes

$$\frac{x^2\,(a^2-v^2)}{a^2-r^2} + \frac{y^2\,(b^2-v^2)}{b^2-r^2} + \frac{z^2\,(c^2-v^2)}{c^2-r^2} = v^2 \;\ldots\ldots\ldots(4);$$

also, we have

$$x^2 + y^2 + z^2 = r^2 \;\ldots\ldots\ldots(5).$$

Subtracting (4) from (5) we get

$$\left\{\frac{x^2}{a^2-r^2} + \frac{y^2}{b^2-r^2} + \frac{z^2}{c^2-r^2}\right\}(v^2-r^2) = -\,(v^2-r^2),$$

$$\text{or} \quad \frac{x^2}{a^2-r^2} + \frac{y^2}{b^2-r^2} + \frac{z^2}{c^2-r^2} + 1 = 0 \;\ldots\ldots\ldots(6),$$

which is one form of the equation to the wave surface. The form used by Fresnel may be easily deduced by combining equations (4) and (6). For if we split each term of (4), it becomes

$$\frac{a^2x^2}{a^2-r^2} + \frac{b^2y^2}{b^2-r^2} + \frac{c^2z^2}{c^2-r^2} - v^2\left\{\frac{x^2}{a^2-r^2} + \frac{y^2}{b^2-r^2} + \frac{z^2}{c^2-r^2}\right\} = v^2,$$

which by the condition (6) reduces itself to

$$\frac{a^2 x^2}{a^2-r^2} + \frac{b^2 y^2}{b^2-r^2} + \frac{c^2 z^2}{c^2-r^2} = 0;$$

in which, if we substitute $x^2 + y^2 + z^2$ for r^2, and multiply up, we shall obtain Fresnel's form of the equation.

G.A. S.

II.—ON THE EQUATION TO THE TANGENT OF THE ELLIPSE.

In the usual method of finding the Equation to the tangent of an Ellipse, the point of contact is given, so that the required equation involves its coordinates, and if we wish to deduce any general properties of the tangent, independent of the particular point of contact, we have to eliminate two quantities, which necessarily renders the operation of elimination troublesome. This may be avoided by finding the condition that a line should be a tangent to an ellipse without specifying the point of contact. For by this means only one indeterminate quantity is introduced, the elimination of which is generally easily effected. The quantity chosen is the tangent of the angle which the tangent to the curve makes with the axis of x.

Let $$y = \alpha x + \beta \quad \text{.........(1)}$$

be the equation to a line cutting the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.........(2).}$$

If we substitute the value of y from (1) in (2) we obtain a quadratic equation in x, the roots of which are the values of x at the two points where the line in general cuts the ellipse. If the line become a tangent, these two values of x must be equal; and on making the condition that the quadratic in x shall be a complete square, we obtain, as an equation of condition,

$$\beta = \sqrt{a^2 \alpha^2 + b^2}.$$

Hence the equation to the tangent to the ellipse may be put under the form

$$y = \alpha x + \sqrt{a^2 \alpha^2 + b^2} \quad \text{......... (3),}$$

where α is the tangent of the angle which the line makes with the axis of x.

This form of the equation is not new, for it is given by Mr. Waud, in p. 73 of his Algebraical Geometry. But that author

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does not seem to have observed its use in deducing several of the principal properties of the Ellipse. Some of these applications we shall now give.

1. To find the locus of the intersection of two tangents to an ellipse, which are at right angles to each other.

By (3), the equation to the one tangent is

$$y = \alpha x + \sqrt{a^2\alpha^2 + b^2};$$

the equation to the other, which is perpendicular to it, is

$$y = -\frac{1}{\alpha}x + \sqrt{\frac{a^2}{\alpha^2} + b^2},$$

$$\text{or} \quad \alpha y = -x + \sqrt{a^2 + b^2\alpha^2}.$$

Transposing, squaring, and adding the two equations, we have

$$(1 + \alpha^2)(x^2 + y^2) = (a^2 + b^2)(1 + \alpha^2),$$

$$\text{whence } x^2 + y^2 = a^2 + b^2.$$

2. To find the product of the perpendiculars from the foci on the tangent.

The coordinates of the foci are

$$x = ae,\ y = 0. \qquad x = -ae,\ y = 0.$$

Therefore the length of the perpendiculars $p_1\ p_2$, on the tangent

$$y = \alpha x + \sqrt{a^2\alpha^2 + b^2},$$

are $p_1 = -\dfrac{\alpha ae + \sqrt{a^2\alpha^2 + b^2}}{\sqrt{1 + \alpha^2}}$, $\qquad p_2 = \dfrac{\alpha ae + \sqrt{a^2\alpha^2 + b^2}}{\sqrt{1 + \alpha^2}}$

therefore

$$p_1 \cdot p_2 = \frac{a^2\alpha^2 + b^2 - a^2e^2\alpha^2}{1 + \alpha^2} = \frac{a^2\alpha^2 + b^2 - a^2\alpha^2 + b^2\alpha^2}{1 + \alpha^2} = b^2.$$

3. To find the locus of the extremity of the perpendiculars from the foci on the tangent.

The equation to the tangent is

$$y = \alpha x + \sqrt{a^2\alpha^2 + b^2}.$$

The equation to the perpendicular on it from the focus is

$$y = -\frac{1}{\alpha}(x - ae),$$

$$\text{or } \alpha y = -x + \sqrt{a^2 - b^2}.$$

Transposing, squaring, and adding, we have

$$(1 + \alpha^2)(x^2 + y^2) = (1 + \alpha^2)a^2, \quad \text{or} \quad x^2 + y^2 = a^2.$$

We shall not proceed to show how by the same means we can find the locus of the perpendicular from the centre on the tangent, or how to prove that $AT . at = b^2$ (*Ham. Con. Sec.* p. 108). as our readers can easily do so for themselves.

The analogous properties of the hyperbola may be proved in the same way by the use of the equation

$$y = ax + \sqrt{a^2 \alpha^2 - b^2},$$

and those of the parabola from the equation

$$y = ax + \frac{m}{a}.$$

But we need do no more than indicate this, as the method is the same as in the ellipse.

O.

III.—ON GENERAL DIFFERENTIATION.

The idea of differential coefficients with general indices is not modern, for it occurred to Leibnitz, who has expressed it in his correspondence with Jean Bernouilli. Euler has written a few pages on this subject, which Lacroix has copied into his large work on the differential calculus. Formulæ for expressing the general differential coefficients of functions by means of definite integrals, have been given by Laplace (*Théorie des Probabilités*, p. 85, 3rd edit.), by Fourier (*Théorie de la Chaleur*, p. 561), and by Mr. Murphy (*Cambridge Phil. Trans.*, vol. 5.). But it appears that the only person who has attempted to reduce the subject to a system, is M. Joseph Liouville; three memoirs by whom,—one on the principles of the calculus, and two on applications of it,—are inserted in the 13th volume of the *Journal de l'Ecole Polytechnique*, for the year 1832. Professor Peacock, in his valuable and interesting Report on certain branches of Analysis, which forms a part of the *Report of the British Association for* 1833, has spoken of M. Liouville's system as erroneous in many essential points, and has given a sketch of one very different. But after referring to M. Liouville's memoirs, and bestowing considerable attention on the subject, we have come to a contrary opinion, at least with respect to his conclusions, which are the same for the most part as will be found in this article. Some points in his theory we admit to be objectionable, and these we have altered.

2. The transition from differential coefficients whose indices are positive integers, to those whose indices are any whatever, should be made in the same manner as the transition in algebra, from symbols of quantity with positive integral indices to those with general indices. Before we can prove any equations involving $\frac{d^\alpha u}{dx^\alpha}$, where α is general, we must affix a meaning to that expression, which can only be done by making some definition or assumption

respecting it. The assumption ought to be such that our results may coincide with the known results of the differential calculus when a becomes a positive integer. We shall therefore assume that the following equations, proved for differential coefficients with positive integral indices, hold true for differential coefficients with general indices:

$$\frac{d^\alpha (u+v)}{dx^\alpha} = \frac{d^\alpha u}{dx^\alpha} + \frac{d^\alpha v}{dx^\alpha} \text{........(A)},$$

$$\frac{d^\alpha}{dx^\alpha} \cdot \frac{d^\beta}{dx^\beta} u = \frac{d^{\alpha+\beta} u}{dx^{\alpha+\beta}} \text{..............(B)},$$

$$\frac{d^\alpha}{dx^\alpha} \cdot \frac{d^\beta u}{dy^\beta} = \frac{d^\beta}{dy^\beta} \cdot \frac{d^\alpha u}{dx^\alpha} \text{...........(C)}.$$

3. From equation (A) it follows that if a be a constant,

$$\frac{d^\alpha . au}{dx^\alpha} = a \frac{d^\alpha u}{dx^\alpha}.$$

When a is a positive integer, this is very easily proved by making $v = u,\ 2u,\ 3u, \ldots\ldots$ to $(a-1)u$, in succession. Next, let $a = \frac{p}{q}$, p and q being positive integers. Then, by the former case,

$$q \frac{d^\alpha . \frac{p}{q} u}{dx^\alpha} = \frac{d^\alpha . q . \frac{p}{q} u}{dx^\alpha} = \frac{d^\alpha . pu}{dx^\alpha} = p \frac{d^\alpha u}{dx^\alpha};$$

therefore

$$\frac{d^\alpha . \frac{p}{q} u}{dx^\alpha} = \frac{p}{q} \frac{d^\alpha u}{dx^\alpha}.$$

The proposition is also easily proved for a negative constant by assuming, in equation (A), $v = -u$. It seems that it cannot be proved when a is not a real quantity, but we shall extend the proposition to this case by assumption.

4. From equation (B) it may be easily deduced that

$$\left(\frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}}\right)^q u = \frac{d^p u}{dx^p}.$$

that is, that the operation denoted by $\frac{d^{\frac{p}{q}}}{dx^{\frac{p}{q}}}$ is such, that being performed q times in succession upon u, the result will be $\frac{d^p u}{dx^p}$. The same equation also enables us to interpret the meaning of $\frac{d^{-n} u}{dx^{-n}}$, n being an integer, for by making $\alpha = -n$, and $\beta = n$, in equation (B), we have

$$\frac{d^{-n}}{dx^{-n}}\cdot\frac{d^n u}{dx^n} = \frac{d^0 u}{dx^0} = u;$$

whence it follows that $\dfrac{d^{-n}}{dx^{-n}}$ is the inverse of the operation $\dfrac{d^n}{dx^n}$.

But we know that the inverse of $\dfrac{d^n}{dx^n}$ is the n^{th} integral with respect to x, therefore $\dfrac{d^{-n}}{dx^{-n}}$ denotes the n^{th} integral with respect to x.

5. The most important conclusion to be deduced from (C) is obtained by supposing $\beta = -1$, whence

$$\frac{d^\alpha}{dx^\alpha}\int u\,dy = \int\frac{d^\alpha u}{dx^\alpha}\,dy.$$

In deducing this formula the integral has been supposed indefinite: but it is easy to see that differentiation, under the sign of integration, is also allowable when the integral is taken between limits, provided that neither of the limiting values of y involve the parameter x.

6. Since, as has been shown in § 4, general differentiation includes integration as a particular case, and since the complete expression of an integral involves arbitrary constants, it follows that the complete expression of a general differential coefficient must involve arbitrary constants. We may express this by saying that the general value of $\dfrac{d^\alpha 0}{dx^\alpha}$ is not 0. It is evident also, that we may introduce the proper arbitrary constants into any expression involving $\dfrac{d^\alpha u}{dx^\alpha}$, by adding to this quantity $\dfrac{d^\alpha 0}{dx^\alpha}$, which we shall call with Liouville the *complementary function.* We may therefore neglect the complementary function in investigating general formulæ. Its form will be investigated hereafter.

7. We proceed to investigate the values of the general differential coefficients of various simple functions, and shall begin with ε^{nx}, because the result is easiest to be obtained, and may be made the foundation of all the rest of the calculus.

Put $$y = \varepsilon^{nx},$$

then y satisfies the equation

$$\frac{dy}{dx} - ny = 0.$$

Hence by equation (A) and § (3)

$$\frac{d^\alpha}{dx^\alpha}\frac{dy}{dx} - n\frac{d^\alpha y}{dx^\alpha} = 0.$$

By equation (B) $\dfrac{d^\alpha}{dx^\alpha}\dfrac{dy}{dx} = \dfrac{d^{\alpha+1}y}{dx^{\alpha+1}} = \dfrac{d}{dx}\dfrac{d^\alpha y}{dx^\alpha}$,

$$\text{therefore } \frac{d}{dx}\cdot\frac{d^\alpha y}{dx^\alpha} - n\,\frac{d^\alpha y}{dx^\alpha} = 0,$$

$$\text{whence } \frac{d^\alpha y}{dx^\alpha} = \mathrm{C}\,\varepsilon^{nx}.$$

Let $a = \frac{p}{q}$, then, by § (4),

$$\mathrm{C}^q\,\varepsilon^{nx} = \frac{d^p\varepsilon^{nx}}{dx^p} = n^p\,\varepsilon^{nx},$$

whether p be positive or negative. Hence $\mathrm{C} = n^{\frac{q}{p}}$, and

$$\frac{d^a\varepsilon^{nx}}{dx^\alpha} = n^\alpha\varepsilon^{nx}\ \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{D}).$$

8. Since, if x be positive,

$$\frac{1}{x} = \int_0^\infty \varepsilon^{-\gamma x} d\gamma,$$

$$\frac{d^\alpha}{dx^\alpha}\frac{1}{x} = \int_0^\infty \varepsilon^{-\gamma x}\,(-\gamma)^\alpha d\gamma.$$

Let $\gamma\, x = \theta$, then the integral is changed to

$$(-1)^\alpha.\ \frac{1}{x^{1+\alpha}}.\int_0^\infty \varepsilon^{-\theta}\theta^a d\theta.$$

If we designate, as Legendre has done, the definite integral

$$\int_0^\infty \varepsilon^{-\theta}\,\theta^{\alpha-1}\,d\theta, \text{ by } \Gamma(a),$$

we have then

$$\frac{d^\alpha}{dx^\alpha}\frac{1}{x} = (-1)^\alpha.\,\Gamma(1+a).\,\frac{1}{x^{1+a}}\ldots\ldots\ldots(\mathrm{E}).$$

We have supposed x positive, but since the formula is not altered after changing x into $-x$ and reducing, it holds whether x be positive or negative.

9. To find the value of $\frac{d^\alpha}{dx^\alpha}\cdot\frac{1}{x^n}$ we proceed as follows;

Supposing x positive, and making $\gamma x = \theta$ in the definite integral

$$\int_0^\infty \varepsilon^{-\gamma x}\,\gamma^{n-1}\,d\gamma,$$

we find,

$$\int_0^\infty \varepsilon^{-\gamma x}\,\gamma^{n-1}\,d\gamma = \frac{1}{x^n}.\int_0^\infty \varepsilon^{-\theta}\,\theta^{n-1}\,d\theta = \Gamma(n)\,.\,\frac{1}{x^n}.$$

Therefore

$$\frac{1}{x^n} = \frac{\int_0^\infty \varepsilon^{-\gamma x}\,\gamma^{n-1}\,d\gamma}{\Gamma(n)},$$

$$\frac{d^\alpha}{dx^\alpha}\frac{1}{x^n} = \frac{\int_0^\infty \varepsilon^{-\gamma x}\,(-\gamma)^\alpha\,\gamma^{n-1}\,d\gamma}{\Gamma(n)};$$

and by making $\gamma x = \theta$,

$$\frac{d^\alpha}{dx^\alpha}\frac{1}{x^n} = (-1)^\alpha . \frac{\Gamma(n+\alpha)}{\Gamma(n)} . \frac{1}{x^{n+\alpha}}.$$

(7)

The remark at the end of the last section applies equally to this expression.

10. We shall digress to prove a few of the most important properties of the definite integral $\Gamma(n)$. It is the second of the Eulerian integrals, as Legendre has called them, from their having been first treated of by Euler. It may be put into other forms beside $\int_0^\infty \varepsilon^{-\theta}\,\theta^{n-1}\,d\theta$, for by making $\varepsilon^{-\theta} = x$, this becomes

$$\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx;$$

again, by making $\theta^n = z$, the former expression becomes

$$\frac{1}{n}\int_0^\infty \varepsilon^{-z^{\frac{1}{n}}}\,dz.$$

11. The first of the properties of the integral in question is, that if n be positive,

$$\Gamma(1+n) = n\,\Gamma(n) \ldots\ldots\ldots (G).$$

For, integrating by parts,

$$\int \varepsilon^{-\theta}\,\theta^n\,d\theta = -\,\varepsilon^{-\theta}\,\theta^n + n\int \varepsilon^{-\theta}\,\theta^{n-1}\,d\theta,$$

Now the integrated part vanishes when $\theta = \infty$, and also when $\theta = 0$, provided n be positive; hence, in this case,

$$\Gamma(1+n) = n\,\Gamma(n).$$

If $n = 0$, the integrated part becomes -1 when $\theta = 0$; and if n be negative, it becomes ∞; so that in neither of these cases the equation (G) is true.

It follows from equation (G) that if r be any integer less than $1+n$,

$$\frac{\Gamma(1+n)}{\Gamma(1+n-r)} = n\,(n-1)\,(n-2)\ldots(n-r+1) \ldots\ldots\ldots (H).$$

If n be a positive integer we may make $r = n$; whence, observing that $\Gamma(1) = \int_0^\infty \varepsilon^{-\theta}\,d\theta = 1$, we find

$$\Gamma(1+n) = n\,(n-1)\,(n-2)\ldots\ldots 2.1 \ldots\ldots\ldots (I).$$

12. It is desirable to examine what $\Gamma(n)$ becomes when n is 0 or negative. For this purpose, let it be observed that the integral

$$\int_0^\infty \varepsilon^{-\theta}\,\theta^{n-1}\,d\theta$$

is equivalent to the infinite series

$$\varepsilon^{-d\theta}\,(d\theta)^{n-1}\,d\theta + \varepsilon^{-2d\theta}\,(2d\theta)^{n-1}\,d\theta + \varepsilon^{-3d\theta}\,(3d\theta)^{n-1}\,d\theta + \ldots\ldots$$

$$\text{or } \Gamma(n) = \{\varepsilon^{-d\theta}.\,1^{n-1} + \varepsilon^{-2d\theta}.\,2^{n-1} + \varepsilon^{-3d\theta}.\,3^{n-1} + \ldots\ldots\}\,(d\theta)^n.$$

Now as long as n is positive, the sum of the series within the brackets is infinite, but it is multiplied by the infinitely small quantity $(d\theta)^n$; there is no reason therefore why the value of the

expression should not be finite. When $n = 0$, the above expression for $\Gamma(n)$ becomes

$$-\log(1-\varepsilon^{-d\theta}) = -\log 0 = \infty.$$

When n is negative, the series within the brackets becomes finite, but it is multiplied by the infinite quantity $(d\theta)^n$, therefore $\Gamma(n)$ is infinite when n is negative. It may be remarked that though $\Gamma(0)$ and $\Gamma(-n)$ are both infinite, $\dfrac{\Gamma(0)}{\Gamma(-n)}$ is 0, because $(d\theta)^n \log(1-\varepsilon^{-d\theta})$ is 0, however small n may be.

13. To prove that

$$\frac{\Gamma(m).\,\Gamma(n)}{\Gamma(m+n)} = \left(\frac{m}{n}\right) \ldots\ldots\ldots\ldots (\text{K}),$$

where $\left(\dfrac{m}{n}\right)$ denotes $\int_0^1 (1-x)^{m-1} x^{n-1} dx$, which is called the first Eulerian integral.

We have

$$\Gamma(m).\,\Gamma(n) = \int_0^\infty \varepsilon^{-x} x^{m-1} dx \,.\int_0^\infty \varepsilon^{-y} y^{n-1} dy$$
$$= \int_0^\infty \int_0^\infty \varepsilon^{-x-y} x^{m-1} y^{n-1} dx\, dy\,;$$

change y into xy, and dy into xdy, then the last expression becomes

$$\int_0^\infty \int_0^\infty \varepsilon^{-x(1+y)} x^{m+n-1} y^{n-1} dx\, dy.$$

Change x into $\dfrac{x}{1+y}$, and dx into $\dfrac{dx}{1+y}$, then this becomes

$$\int_0^\infty \int_0^\infty \varepsilon^{-x} x^{m+n-1} \frac{y^{n-1}}{(1+y)^{m+n}} dx\, dy$$
$$= \int_0^\infty \varepsilon^{-x} x^{m+n-1} dx \,. \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy.$$

The first factor is $\Gamma(m+n)$; the second is to be transformed as follows. Assume $1+y = \dfrac{1}{1-z}$, then $dy = \dfrac{dz}{(1-z)^2}$, and $y = \dfrac{z}{1-z}$; when $y = 0$, $z = 0$, and when $y = \infty$, $z = 1$. Hence

$$\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^1 (1-z)^{m-1} z^{n-1} dz = \left(\frac{m}{n}\right);$$

and the proposition is manifest.

Supposing m and n, which are in general fractions, to be reduced to a common denominator, and to be equal to $\dfrac{q}{r}, \dfrac{p}{r}$, and changing z into x^r, the integral

$$\int_0^1 (1-z)^{m-1} z^{n-1} dz \text{ becomes } r\int_0^1 \frac{x^{p-1} dx}{(1-x^r)^{1-\frac{q}{r}}},$$

which is the form adopted by several writers. A good many of its properties are proved in Hymers's Integral Calculus.

14. Let $n = 1 - m$, then $\Gamma(m+n)$ becomes 1, and the equation just proved becomes

$$\Gamma(m)\,\Gamma(1-m) = \int_0^1 (1-z)^{-1+m}\, z^{-m}\, dz;$$

the value of which integral is $\frac{\pi}{\sin m\pi}$, if m be between 0 and 1.

The following is a new method of finding it:

Let $z = (\sin\theta)^2$, then $1 - z = (\cos\theta)^2$, $dz = 2\sin\theta\cos\theta\, d\theta$, when $z = 0$, $\theta = 0$, and when $z = 1$, $\theta = \frac{\pi}{2}$.

Hence $\int_0^1 (1-z)^{-1+m}\, z^{-m}\, dz = 2\int_0^{\frac{\pi}{2}} (\tan\theta)^{1-2m}\, d\theta.$

If we put for $\tan\theta$ its value $\frac{1}{\sqrt{-1}}\,\frac{1-\varepsilon^{-\sqrt{-1}\,2\theta}}{1+\varepsilon^{-\sqrt{-1}\,2\theta}}$, it is evident that $\left(\frac{1-\varepsilon^{-\sqrt{-1}\,2\theta}}{1+\varepsilon^{-\sqrt{-1}\,2\theta}}\right)^{1-2m}$ may be expanded in the form

$$\begin{aligned} & 1 + A_1\,\varepsilon^{-\sqrt{-1}\,2\theta} + A_2\,\varepsilon^{-\sqrt{-1}\,4\theta} + \ldots\ldots \\ &= 1 + A_1 \cos 2\theta + A_2 \cos 4\theta + \ldots\ldots \\ &\quad -\sqrt{-1}\,(A_1 \sin 2\theta + A_2 \sin 4\theta + \ldots\ldots). \end{aligned}$$

Also, $(\sqrt{-1})^{1-2m} = \cos(1-2m)\frac{\pi}{2} + \sqrt{-1}\sin(1-2m)\frac{\pi}{2}$

$$= \sin m\pi + \sqrt{-1}\cos m\pi.$$

Substituting the series, multiplying by the above value of $(\sqrt{-1})^{1-2m}$, and equating real parts, we find

$$\sin m\pi \int_0^{\frac{\pi}{2}} (\tan\theta)^{1-2m} d\theta = \int_0^{\frac{\pi}{2}} (1 + A_1\cos 2\theta + A_2 \cos 4\theta + \ldots)\, d\theta = \frac{\pi}{2}$$

since the periodic terms vanish at each limit. Hence

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi} \quad \ldots\ldots\ldots\ldots (L).$$

It is necessary that m should be less than 1, otherwise $(\tan\theta)^{1-2m} d\theta$ would not be infinitely small when θ is so.

15. In equation (L), put $m = \frac{1}{2}$, therefore $\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi$, and

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

Hence, by equation (G), § 11, we find

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3.1}{2^2}\sqrt{\pi}, \ \ldots\ldots$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{2^n}\sqrt{\pi}.$$

16. We now return to our subject of general differentiation. Our readers will find no difficulty in applying equation (H) to obtain from formula (F) the known values of $\frac{d^\alpha}{dx^\alpha}\frac{1}{x^n}$ when a is an integer, positive or negative, provided that $n + a$ be positive. But that formula must not be extended to cases where n or $n + a$ is negative. For it depends upon the equation

$$\frac{1}{x^r} = \frac{\int_0^\infty \varepsilon^{-\gamma x}\,\gamma^{r-1}\,d\gamma}{\Gamma(r)};$$

which represents the developement of $\frac{1}{x^r}$ in a series of exponentials; and the infinite value of $\Gamma(r)$ when r is negative shews that a positive power of x cannot be developed in such a series. We cannot therefore rely upon results which are obtained by supposing this equation true for negative values of r. But, without supposing formula (F) to hold when the index of x on either side is positive, we may deduce from it formulæ to suit such cases.

17. To find $\frac{d^\alpha x^n}{dx^\alpha}$ when n is positive and $n - a$ negative.

By formula (F)

$$\frac{d^\alpha}{dx^\alpha}\frac{1}{x^n} = (-1)^\alpha \cdot \frac{\Gamma(n+a)}{\Gamma(n)} \cdot \frac{1}{x^{n+\alpha}};$$

where we suppose, for convenience, n to be between 0 and 1. Integrate both sides p times, p being any number less than $n + a$; then the first side becomes

$$\frac{d^\alpha}{dx^\alpha}\int^p \frac{1}{x^n}\,dx^p = \frac{1}{(1-n)(2-n)\,..\,(p-n)}\cdot\frac{d^\alpha}{dx^\alpha}x^{p-n}$$

$$= \frac{\Gamma(1-n)}{\Gamma(1+p-n)}\cdot\frac{d^\alpha}{dx^\alpha}\cdot x^{p-n};$$

and the second side becomes

$$(-1)^\alpha \cdot \frac{\Gamma(n+a)}{\Gamma(n)} \cdot (-1)^p \cdot \frac{1}{(n+a-1)(n+a-2)\,..(n+a-p)}\cdot\frac{1}{x^{n+\alpha-p}}$$

$$= (-1)^{\alpha+p}\,\frac{\Gamma(n+a-p)}{\Gamma(n)}\;\frac{1}{x^{n+\alpha-p}};$$

wherefore

$$\frac{d^\alpha}{dx^\alpha}x^{p-n} = (-1)^{\alpha+p}\cdot\frac{\Gamma(1+p-n)\,\Gamma(a-p+n)}{\Gamma(n)\,\Gamma(1-n)}\cdot\frac{1}{x^{\alpha-p+n}};$$

but since n is between 0 and 1, by equation (L),

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} = (-1)^{p-1}\,\frac{\pi}{\sin(p-n)\pi}.$$

Making this substitution, and changing $p-n$ into n,

$$\frac{d^\alpha x^n}{dx^\alpha} = (-1)^{\alpha+1} \frac{\sin n\pi}{\pi} \cdot \Gamma(1+n) . \Gamma(\alpha-n) \cdot \frac{1}{x^{\alpha-n}} \ldots\ldots (\text{M}).$$

18. From this formula we may immediately deduce one for the contrary case, namely that where the index of x is negative before differentiation and positive afterwards, the index of differentiation being negative. For, affecting both sides of the last equation with $\frac{d^{-\alpha}}{dx^{-\alpha}}$, and dividing by the constants,

$$\frac{d^{-\alpha}}{dx^{-\alpha}} \frac{1}{x^{\alpha-n}} = (-1)^{\alpha+1} . \frac{\pi}{\sin n\pi . \Gamma(1+n)\Gamma(\alpha-n)} \cdot x^n ;$$

and changing $\alpha - n$ into n, and therefore n into $\alpha - n$,

$$\frac{d^{-\alpha}}{dx^{-\alpha}} \frac{1}{x^n} = (-1)^{\alpha+1} . \frac{\pi}{\sin(\alpha-n)\pi . \Gamma(1+\alpha-n)\Gamma(n)} \cdot x^{\alpha-n} \ldots\ldots (\text{N}).$$

19. It remains to find a formula for the case where the index of x is positive both before and after differentiation. For this purpose suppose $\alpha-n$ to be between 0 and 1 in formula (M), then the first side becomes

$$\frac{1}{(n+1)(n+2)\ldots(n+p)} . \frac{d^\alpha x^{n+p}}{dx^\alpha} = \frac{\Gamma(1+n)}{\Gamma(1+n+p)} . \frac{d^\alpha x^{n+p}}{dx^\alpha} ;$$

and on the second side the index of x will become $p+n-\alpha$, and it will be multiplied by

$$\frac{1}{(1-\alpha+n)(2-\alpha+n)\ldots(p-\alpha+n)} = \frac{\Gamma(1-\alpha+n)}{\Gamma(1+n+p-\alpha)} .$$

Therefore

$$\frac{d^\alpha x^{n+p}}{dx^\alpha} = (-1)^{\alpha+1} . \frac{\sin n\pi}{\pi} \times$$

$$\Gamma(\alpha-n) . \Gamma(1-\alpha+n) . \frac{\Gamma(1+n+p)}{\Gamma(1+n+p-\alpha)} \cdot x^{p+n-\alpha}.$$

Now

$$\Gamma(\alpha-n) . \Gamma(1-\alpha+n) = \frac{\pi}{\sin(\alpha-n)\pi} = \frac{\pi}{(-1)^{p+1}\sin(n+p-\alpha)\pi},$$

and $\sin n\pi = (-1)^p \sin(n+p)\pi$. Substituting, and changing $n+p$ into n, we find

$$\frac{d^\alpha x^n}{dx^\alpha} = (-1)^\alpha . \frac{\sin n\pi}{\sin(n-\alpha)\pi} \cdot \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} \cdot x^{n-\alpha} \ldots\ldots (\text{O}).$$

This formula may be easily shewn, by the same process as in § 18, to be true when α is negative.

The factor in the last expression, $(-1)^\alpha \frac{\sin n\pi}{\sin(n-\alpha)\pi}$, is remarkable for becoming equal to unity whenever α is an integer, while it admits of any value between $+\infty$ and $-\infty$ when α is fractional.

20. The most important features in the formulæ just investigated are the sines of multiples of the semicircumference. From formulæ (M) and (Q) it follows that the differentials to fractional indices of positive integral powers of x are nothing; and from formulæ (N) and (O), that when the index of x is fractional before differentiation, and a positive integer after it, the differential coefficient is infinite. It is true that in the investigations we excluded the cases of the index of x being integral before or after differentiation, but that was only for convenience. In order to find the results in those cases, it would be necessary, where we supposed the index of x to be between 0 and -1, instead of that to suppose it equal to -1, and by considering that $\frac{x^0}{0}$, as a value of $\int \frac{dx}{x}$, is not false, but only differs from the value $\log x$ by an infinite arbitrary constant, and that our object here is to get the value of $\frac{d^\alpha x^n}{dx^\alpha}$ in the form $\mathrm{M}\, x^{n-\alpha}$; we shall see that our formulæ (M), (N), (O), give the right results in the above mentioned cases. The only restriction upon the generality of the formulæ is that we must not make quantities under the sign Γ negative or nothing.

21. It is desirable to obtain a formula for $\frac{d^\alpha x^n}{dx^\alpha}$ which shall be generally true whatever be the values of a and n. This may be done, though not in terms of the function Γ.

Assume

$$\frac{d^\alpha x^n}{dx^\alpha} = \mathrm{M}\, x^{n-\alpha};$$

and take the $(n-\alpha)^{th}$ differential coefficient of both sides, therefore

$$\frac{d^n x^n}{dx^n} = \mathrm{M}\, \frac{d^{n-\alpha} x^{n-\alpha}}{dx^{n-\alpha}}.$$

The value of $\frac{d^n x^n}{dx^n}$ is independent of x; let it be represented by $\mathrm{P}(n)$, then we have

$$\mathrm{M} = \frac{\mathrm{P}(n)}{\mathrm{P}(n-\alpha)},$$

and

$$\frac{d^\alpha x^n}{dx^\alpha} = \frac{\mathrm{P}(n)}{\mathrm{P}(n-\alpha)} \cdot x^{n-\alpha} \text{.........(P)}.$$

It appears from formulæ (N) and (O), that the value of $\mathrm{P}(n)$ is infinite in every case except when n is a positive integer, in which case it becomes $1.\,2.\,3 \ldots . n$. It will in all cases possess the property

$$\mathrm{P}(n) = n\,\mathrm{P}(n-1).$$

22. We are now enabled to assign the form of the complementary function. The quantity which is to be added to the a^{th} differential coefficient of any function to render it complete, must evidently be one of which the $(-a)^{th}$ differential coefficient is nothing. But we have seen in § 20, that the fractional differential coefficient of a power of x vanishes when the index of that power is a positive integer, and in no other case; consequently the form of the complementary function is

$$C_0 + C_1 x + C_2 x^2 + \ldots\ldots$$

the number of terms being indefinite when the index of differentiation is a fraction.

S. S.

IV.—ON A PROPERTY OF THE TRIANGLE.

The following property of a triangle is remarkable not only for the curious relation between certain lines, but for its leading readily to two elegant theorems regarding the radii of the circles which touch the three sides of a triangle.

Let x, y, z be the perpendiculars from any point on the sides of a triangle, p, q, r the perpendiculars respectively parallel to them through the angles. Then

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

If we join the point, from which the perpendiculars are drawn, with the three angles, the whole triangle will be divided into three triangles, which have the sides of the triangles as bases, and the lines x, y, z as their vertical heights. Let a, b, c be the sides of the triangle, then $\frac{ax}{2}, \frac{by}{2}, \frac{cz}{2}$ will be the areas of the three parts;

and $\dfrac{ax + by + cz}{2}$ = area of whole triangle.

Also area of whole triangle $= \dfrac{ap}{2} = \dfrac{bq}{2} = \dfrac{cr}{2}$.

Dividing the corresponding terms by these quantities we get

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

This relation between the lines is evidently that of the coordinates of a plane which cuts the axes at points whose distances from the origin are p, q, r.

If the point were outside of the triangle, we should have to subtract one of the terms, such as $c\,z$, so that the resulting equation would be

$$\frac{x}{p} + \frac{y}{q} - \frac{z}{r} = 1.$$

Now let ρ be the radius of the inscribed circle, then taking the centre of this circle as the given point, we have $x = y = z = \rho$, and consequently

$$\frac{1}{\rho} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Again, let ρ_1, ρ_2, ρ_3 be the radii of the circles which touch one of the sides of the triangle externally, and the other two internally; then we shall have, by similar reasoning,

$$\frac{1}{\rho_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r}$$

$$\frac{1}{\rho_2} = \frac{1}{p} - \frac{1}{q} + \frac{1}{r}$$

$$\frac{1}{\rho_3} = -\frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Adding these equations together, we get

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{\rho}.$$

It is obvious that similar theorems hold good for any tetrahedron, but it is needless to do more than indicate them.

W. W.

V.—ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

The following method of integrating linear differential equations deserves attention, not only as leading readily to the solution of these equations, but also as placing their theory in a clear light, and pointing out the cause of the success of the method usually employed.

M. Brisson appears to have been the first person who applied the principle of the separation of the signs of operation from those of quantity to the solution of differential equations. This he did in two memoirs of the dates of 1821 and 1823, but we have not been fortunate enough to meet with them, (if indeed they have been published), and our knowledge of them is derived from a

casual notice in a memoir of Cauchy on the same subject, in his *Exercices*, vol. ii. p. 159. This last author seems to have pursued a different course from Brisson; and as it does not appear to be the best for putting the subject in a clear light, we have taken the liberty of deviating very considerably from his method, and in so doing we have probably approached nearer to that of Brisson.

If we take the general linear equation with constant coefficients

$$\frac{d^n y}{dx^n} + \mathrm{A}\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{B}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + \mathrm{R}\frac{dy}{dx} + \mathrm{S}y = \mathrm{X},$$

when X is any function of x, and separate the signs of operation from those of quantity, it becomes

$$\left(\frac{d^n}{dx^n} + \mathrm{A}\frac{d^{n-1}}{dx^{n-1}} + \mathrm{B}\frac{d^{n-2}}{dx^{n-2}} + \ldots\ldots + \mathrm{R}\frac{d}{dx} + \mathrm{S}\right)y = \mathrm{X}.$$

The quantity within the brackets involving only constants, and the signs of operation may be considered as one operation performed on y, and it may be represented by

$$f\left(\frac{d}{dx}\right)y = \mathrm{X}.$$

Here y is given at once explicitly if we are able to perform the inverse operation of $f\left(\frac{d}{dx}\right)$. For if we represent the inverse operation by the usual symbol $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$, and perform that operation on both sides, we get

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}\cdot f\left(\frac{d}{dx}\right)y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1}\mathrm{X},$$

or,

$$y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1}\mathrm{X}.$$

It is plain that its general form we cannot easily perform the inverse operation $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$; but if we begin with a simple case we shall be easily led to a means of effecting it.

Let us take the equation

$$y + \frac{dy}{dx} = ax^n,$$

or,

$$\left(1 + \frac{d}{dx}\right)y = ax^n.$$

Now the inverse operation of $\left(1 + \frac{d}{dx}\right)$ is $\left(1 + \frac{d}{dx}\right)^{-1}$. Therefore

$$y = \left(1 + \frac{d}{dx}\right)^{-1} ax^n.$$

But as in integration there must be added an arbitrary constant which vanishes by differentiation, so here we must add a function which will vanish when the operation $\left(1+\frac{d}{dx}\right)$ is performed on it. This complementary function may be found from that condition, but the following more direct method is perhaps preferable. Since the result of the operation $1+\frac{d}{dx}$ on the function is 0, we may put the value of y under the form

$$y=\left(1+\frac{d}{dx}\right)^{-1}ax^n+\left(1+\frac{d}{dx}\right)^{-1}0.$$

Now if we treat the symbols of operation as if they were symbols of quantity, we have

$$\left(1+\frac{d}{dx}\right)^{-1}0=\frac{d^{-1}}{dx^{-1}}\left(1+\frac{d^{-1}}{dx^{-1}}\right)^{-1}0.$$

But $\frac{d^{-1}}{dx^{-1}}$ is the same as $\int dx$. Hence

$$\left(1+\frac{d}{dx}\right)^{-1}0=\left(1+\frac{d^{-1}}{dx^{-1}}\right)^{-1}\mathrm{C},$$

(C being the arbitrary constant arising from the integration)

$$=\left(1-\frac{d^{-1}}{dx^{-1}}+\frac{d^{-2}}{dx^{-1}}-\dots\right)\mathrm{C};$$

or, performing the operations indicated,

$$=\mathrm{C}\left(1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\dots\right)=\mathrm{C}\,\varepsilon^{-x}.$$

$$\text{Hence } y=\left(1+\frac{d}{dx}\right)^{-1}a\,x^n+\mathrm{C}\,\varepsilon^{-x}.$$

Now expanding the first term

$$y=\left(1-\frac{d}{dx}+\frac{d^2}{dx^2}-\frac{d^3}{dx^3}+\dots\right)a\,x^n+\mathrm{C}\,\varepsilon^{-x}.$$

Therefore

$$y=a\left(x^n-n\,x^{n-1}+n.\overline{n-1}\,x^{n-2}-\dots\dots\right)+\mathrm{C}\,\varepsilon^{-x}.$$

As the operation $\left(1+\frac{d}{dx}\right)^{-1}$ frequently occurs in these equations it is convenient to recollect that we must always add the function $\mathrm{C}\,\varepsilon^{-x}$. And in the same way it would be seen if the operation be $\left(a+\frac{d}{dx}\right)^{-1}$ the complementary function is $\mathrm{C}\,\varepsilon^{-ax}$, and similarly for all Binomial symbols of operation of this kind.

Equations of the first degree, when the coefficients of y and $\frac{dy}{dx}$ are functions of x, are easily reduced to this case by a change of the independent variable. Let us take as an example the equation

$$\frac{dy}{dx} + \frac{ny}{\sqrt{1+x^2}} = a,$$

$$\text{or } \left(1 + \frac{\sqrt{1+x^2}}{n}\frac{d}{dx}\right) y = \frac{a\sqrt{1+x^2}}{n}.$$

Let $\frac{n\,dx}{\sqrt{1+x^2}} = dt$, therefore $\frac{t}{n} = \log(x + \sqrt{1+x^2})$;

$$\text{whence } \sqrt{1+x^2} = \tfrac{1}{2}\left(\varepsilon^{\frac{t}{n}} + \varepsilon^{-\frac{t}{n}}\right),$$

and the equation becomes

$$\left(1 + \frac{d}{dt}\right) y = \frac{a}{2n}\left(\varepsilon^{\frac{t}{n}} + \varepsilon^{-\frac{t}{n}}\right);$$

$$\text{therefore } y = \left(1 + \frac{d}{dt}\right)^{-1} \frac{a}{2n}\left(\varepsilon^{\frac{t}{n}} + \varepsilon^{-\frac{t}{n}}\right) + c\varepsilon^{-t};$$

or, expanding the first term,

$$y = \frac{a}{2n}\left(1 - \frac{d}{dt} + \frac{d^2}{dt^2} - \&c.\right)\left(\varepsilon^{\frac{t}{n}} + \varepsilon^{-\frac{t}{n}}\right) + c\varepsilon^{-t}$$

$$= \frac{a}{2n}\left(1 - \frac{1}{n} + \frac{1}{n^2} - \&c.\right)\varepsilon^{\frac{t}{n}}$$

$$+ \frac{a}{2n}\left(1 + \frac{1}{n} + \frac{1}{n^2} - \&c.\right)\varepsilon^{-\frac{t}{n}} + c\varepsilon^{-t};$$

$$\text{therefore } y = \frac{a}{2(n+1)}\varepsilon^{\frac{t}{n}} + \frac{a}{2(n-1)}\varepsilon^{-\frac{t}{n}} + c\varepsilon^{-t},$$

or substituting for t its value in terms of x,

$$y = \frac{a}{2(n+1)}(\sqrt{1+x^2} + x) + \frac{a}{2(n-1)}(\sqrt{1+x^2} - x)$$
$$+ c(\sqrt{1+x^2} + x)^n.$$

It is needless to multiply examples, as the principle of the method in the case of equations of the first order is sufficiently obvious from those given. But we will proceed to prove a theorem which is very useful, particularly in equations of the higher orders. The theorem is, that

$$\left(\frac{d}{dx} \pm a\right)^n X = \varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^n \varepsilon^{\pm ax} X.$$

For, if we expand the first side, we have

$$\left(\frac{d}{dx} \pm a\right)^n X = \left(\frac{d^n}{dx^n} \pm na\frac{d^{n-1}}{dx^{n-1}} + \frac{n.n-1}{1.2}a^2\frac{d^{n-2}}{dx^{n-2}} \pm \&c.\right) X.$$

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Now, $\pm a^p = \varepsilon^{\mp ax} \left(\frac{d}{dx}\right)^p \varepsilon^{\pm ax}$,

so that the second side may be put under the form

$$\varepsilon^{\mp ax}\left(\frac{d^n}{dx^n} + n\frac{d^{n-1}}{dx^{n-1}}\cdot\frac{d'}{dx} + \frac{n.n-1}{1.2}\,\frac{d^{n-2}}{dx^{n-2}}\,\frac{d'^2}{dx^2} + \&c.\right)\varepsilon^{\pm ax}\,\mathrm{X},$$

(where the accented letters refer to $\varepsilon^{\pm ax}$, and the unaccented to X), and this is equivalent to

$$\varepsilon^{\mp ax}\left(\frac{d}{dx} + \frac{d'}{dx}\right)^n \varepsilon^{\pm ax}\,\mathrm{X} = \varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^n \varepsilon^{\pm ax}\,\mathrm{X},$$

by the Theorem of ***Leibnitz***.

When $\mathrm{X} = \varepsilon^{mx}$, the proposition takes the form

$$\left(\frac{d}{dx} \pm a\right)^n \varepsilon^{mx} = (m \pm a)^n\,\varepsilon^{mx}.$$

By this theorem all operations of the nature of $\left(\frac{d}{dx} \pm a\right)^n$ are reduced to differentiation, or, as in the cases to which we have generally to apply it n is negative, to integration.

To return now to the general equation which we represented by

$$f\left(\frac{d}{dx}\right) y = \mathrm{X}.$$

The inverse operation of $f\left(\frac{d}{dx}\right)$ cannot easily be performed directly, but we conceive the operation $f\left(\frac{d}{dx}\right)$ to be made up by the combination of n binomial operations of the form of $\left(\frac{d}{dx} - a\right)$; and, by what we have shown before, we can perform the inverse operation for each of these successively, and this will be equivalent to performing the whole inverse operation of $f\left(\frac{d}{dx}\right)$ at once. For, treating the operation $\frac{d}{dx}$ exactly as if it were a function of x of the same form, we can resolve it into factors, so that it becomes

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right)\&c.\left(\frac{d}{dx} - a_n\right),$$

where a_1, a_2, a_3, &c. are the roots of the equation

$$f(z) = 0.$$

Hence the equation $f\left(\frac{d}{dx}\right) y = \mathrm{X}$ becomes

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right)\&c.\left(\frac{d}{dx} - a_n\right) y = \mathrm{X}.$$

Now, performing the inverse operation of $\left(\frac{d}{dx} - a_1\right)$, we have

$$\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right) y = \left(\frac{d}{dx} - a_1\right)^{-1} X$$
$$= \varepsilon^{a_1 x} \int \varepsilon^{-a_1 x} X\, dx,$$

by the theorem prefixed, since in this case $n = -1$.

We should properly add a term $\left(\frac{d}{dx} - a_1\right)^{-1} 0 = c\varepsilon^{a_1 x}$, but as we may suppose the arbitrary constant to be included in the sign of integration, we may leave out this term for the sake of brevity.

Again, performing the inverse operation of $\left(\frac{d}{dx} - a_2\right)$, we have

$$\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right) y = \left(\frac{d}{dx} - a_2\right)^{-1} \left(\varepsilon^{a_1 x} \int \varepsilon^{-a_1 x} X\, dx\right)$$
$$= \varepsilon^{a_2 x} \int \varepsilon^{(a_1 - a_2)x} \left(\int \varepsilon^{-a_1 x} X\, dx\right) dx.$$

Integrating by parts, this becomes

$$\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right) y = \frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{a_1 - a_2} - \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{a_1 - a_2}$$
$$= \frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{a_1 - a_2} + \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{a_2 - a_1}.$$

Performing the inverse operation of $\left(\frac{d}{dx} - a_3\right)$, we have

$$\left(\frac{d}{dx} - a_4\right) \&c. \left(\frac{d}{dx} - a_n\right) y$$
$$= \left(\frac{d}{dx} - a_3\right)^{-1} \left(\frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{a_1 - a_2} + \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{a_2 - a_1}\right)$$
$$= \frac{\varepsilon^{a_3 x} \int \varepsilon^{(a_1 - a_3)x} \left(\int \varepsilon^{-a_1 x} X dx\right) dx}{a_1 - a_2} + \frac{\varepsilon^{a_3 x} \int \varepsilon^{(a_2 - a_3)x} \left(\int \varepsilon^{-a_2 x} X dx\right)}{a_2 - a_1}.$$

And integrating each of the terms separately by parts, we get, as before,

$$\frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{(a_1 - a_2)(a_1 - a_3)} - \frac{\varepsilon^{a_3 x}\left(\int \varepsilon^{-a_3 x} X\, dx\right)}{(a_1 - a_2)(a_1 - a_3)}$$
$$+ \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{(a_2 - a_1)(a_2 - a_3)} - \frac{\varepsilon^{a_3 x}\left(\int \varepsilon^{-a_3 x} X\, dx\right)}{(a_2 - a_1)(a_2 - a_3)}$$
$$= \frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{(a_1 - a_2)(a_1 - a_3)} + \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{(a_2 - a_1)(a_2 - a_3)} + \frac{\varepsilon^{a_3 x}\left(\int \varepsilon^{-a_3 x} X\, dx\right)}{(a_3 - a_1)(a_3 - a_2)},$$

and so on for every successive factor, so that at last

$$y = \frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X\, dx\right)}{(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n)} + \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X\, dx\right)}{(a_2 - a_1)(a_2 - a_3) \ldots (a_2 - a_n)} + \&c.$$
$$+ \frac{\varepsilon^{a_n x}\left(\int \varepsilon^{-a_n x} X\, dx\right)}{(a_n - a_1)(a_n - a_2) \ldots\ldots (a_n - a_{n-1})}.$$

We shall leave to the reader the application of the general method to particular cases, and shall proceed to show how some equations, of an order higher than the first, may be conveniently solved without operating with each factor separately.

For instance, if we take the example of the equation

$$y + \frac{d^2y}{dx^2} = a \cos mx,$$

$$\text{or } \left(1 + \frac{d^2}{dx^2}\right) y = a \cos mx;$$

$$\text{therefore } y = \left(1 + \frac{d^2}{dx^2}\right)^{-1} a \cos mx + \left(1 + \frac{d^2}{dx^2}\right)^{-1} 0.$$

$$\text{Now, } \left(1 + \frac{d^2}{dx^2}\right)^{-1} 0 = \frac{d^{-2}}{dx^{-2}}\left(1 + \frac{d^{-2}}{dx^{-2}}\right) 0$$

$$= \left(1 + \frac{d^{-2}}{dx^{-2}}\right)(c_1 x + c_2)$$

$$= \left(1 - \frac{d^{-2}}{dx^{-2}} + \frac{d^{-4}}{dx^{-4}} - \&c.\right)(c_1 x + c_2)$$

$$= c_1\left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2....5} - \&c.\right)$$

$$+ c_2\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.\right)$$

$$= c_1 \sin x + c_2 \cos x.$$

$$\text{Also, } \left(1 + \frac{d^2}{dx^2}\right)^{-1} a \cos mx = a\left(1 - \frac{d^2}{dx^2} + \frac{d^4}{dx^4} - \&c.\right) \cos mx$$

$$= a(1 + m^2 + m^4 + \&c.) \cos mx = \frac{a}{1 - m^2} \cos mx;$$

$$\text{whence } y = \frac{a}{1 - m^2} \cos mx + c_1 \sin x + c_2 \cos x.$$

Again, if we have the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2,$$

where the binomial factors of operation are equal, it may be put under the form

$$\left(\frac{d}{dx} - 2\right)^2 y = x^2,$$

$$\text{whence } y = \left(\frac{d}{dx} - 2\right)^{-2} x^2 + \left(\frac{d}{dx} - 2\right)^{-2} 0.$$

$$\text{Now } \left(\frac{d}{dx} - 2\right)^{-2} x^2 = \left(2 - \frac{d}{dx}\right)^{-2} x^2$$

$$= (2^{-2} + 2.2^{-3}\frac{d}{dx} + 3.2^{-4}\frac{d^2}{dx^2} + \&c.)\, x^2 = \frac{x^2}{2^2} + \frac{4x}{2^3} + \frac{6}{2^4}.$$

Also,

$$\left(\frac{d}{dx}-2\right)^{-2}0=\frac{d^{-2}}{dx^{-2}}\left(1-2\frac{d^{-1}}{dx^{-1}}\right)^{-2}0=\left(1-2\frac{d^{-1}}{dx^{-1}}\right)^{-2}(cx+c_1)$$

$$=\left(1+2.2\frac{d^{-1}}{dx^{-1}}+3.2^2\frac{d^{-2}}{dx^{-2}}+4.2^3\frac{d^{-3}}{dx^{-3}}+\&c.\right)(cx+c_1)$$

$$=c\left(x+2.\frac{2x^2}{1.2}+3.\frac{2^2x^3}{1.2.3}+\&c.\right)$$

$$+c_1\left(1+2.2x+3.2^2\frac{2^2}{1.2}+\&c.\right)$$

$$=cx\left(1+2x+\frac{2^2x^2}{1.2}+\frac{2^3x^3}{1.2.3}+\&c.\right)$$

$$+c_1\left(1+2x+\frac{2^2x^2}{1.2}+\&c.\right)+2c_1x\left(1+2x+\frac{2^2x^2}{1.2}+\&c.\right)$$

$$=(c_1+c_2x)\,\varepsilon^{2x},\text{ if } c+2c_1=c_2;$$

therefore we have

$$y=\frac{x^2}{2^2}+\frac{4x}{2^3}+\frac{6}{2^4}+(c_1+c_2x)\,\varepsilon^{2x}.$$

We might have omitted the latter part of this example, as it is easy to show, in the usual manner, what is the form of the complementary function when the two factors are equal, but we preferred the method given, as shewing how we may arrive at the same result directly.

On looking back on the method pursued, it is easy to see the causes of some of the known peculiarities in the usual solution of linear differential equations with constant coefficients. In the first place, their solution is attended with greater facility than that of other differential equations, because in fact y is given *explicitly* at once. In the next place, the exponential function which is assumed for the solution of these equations, is derived from the binomial factors of operation, $\left(a_1-\frac{d}{dx}\right)$ &c.; and as there are n factors in an equation of the n^{th} order, there will be n exponential functions in the complete solution. Lastly, the equation

$$f\left(\frac{d}{dx}\right)y=\text{X}+0$$

may be derived from the equation

$$f\left(\frac{d}{dx}\right)y=0,$$

by differentiation only; for in operating with each factor of the form $\left(\frac{d}{dx}-a\right)^{-1}$ on X, we have only to expand according to powers of $\frac{d}{dx}$, and perform the operations indicated, and then add

a term which must be the same as the term arising from the corresponding operation in the equation

$$f\left(\frac{d}{dx}\right) y = 0.$$

The application of this method to linear differential equations with variable coefficients is attended with considerable difficulty, and indeed neither Brisson nor Cauchy seem to have made any progress in the solution of these equations. There are, however, some which can be thus integrated, but we shall defer to a future number any observations we have to make on them, as well as the application of the same method to equations of finite and mixed differences, in which it is probably more useful than in differential equations.

But, before leaving the subject, we would say a few words on the legitimacy of the processes employed in this method. In the preceding pages we have spoken of treating the symbols of operation like those of quantity, so that at first sight it would appear as if the principles on which the method is founded, were drawn only from analogy. But a little consideration will show that this is not really the case, and that the reasoning on which we proceed is perfectly strict and logical. We have spoken as if there were a distinction between what are usually called symbols of operation, and those which are called symbols of quantity. But we might with perfect propriety call these last also symbols of operation. For instance, x is the operation designated by (x) performed on unity, x^n is the same operation performed n times in succession on unity, $a + x$ is the operation $(a + x)$ performed on unity, $(a + x)^n$ is the operation $(a + x)$ performed n times in succession on unity. By the phrase "in succession" is to be understood, that the operations are performed, so to speak, successively one on the back of the other; and perhaps it would be better to say, that the operation (x) is repeated n times on unity. And in this x^n is to be distinguished from nx, which represents that n of the operations (x) on unity are taken simultaneously. In the same way as $a(1)$ represents the operation (a) performed on (1), $a(x)$ would represent the same operation performed on x, and $a^n(x)$ would represent the operation repeated n times on (x). These operations are usually written ax, $a^n x$.

If, then, we take this view of what are usually called symbols of quantity, we shall have little difficulty in seeing the correctness of the principle by which other operations, such as we represent by $\left(\frac{d}{dx}\right)$, (Δ), &c., are treated in the same way as a, b, &c. For whatever is proved of the latter symbols, from the known laws of their combination, must be equally true of all other symbols which are subject to the same laws of combination. Now the laws of the combinations of the symbols a, b, &c. are, that

$$a^m . a^n x = a^{m+n} . x \text{ ....(1)},$$

$$a\{b(x)\} = b\{a(x)\} \text{ ....(2)},$$

$$\text{and } a(x) + a(y) = a(x + y) \text{......(3)}.$$

And, if f, f_1, &c. be any other general symbols of operation (f and f_1 being of the same kind) subject to the same laws of combination, so that

$$f^m . f^n(x) = f^{(m+n)}(x). \quad \text{........}(1),$$

$$f\{f_1(x)\} = f_1\{f(x)\} \quad \text{...........}(2),$$

$$\text{and } f(x) + f(y) = f(x + y) \text{........}(3).$$

Then, whatever we may have proved of a, b, &c. depending on these three laws, must necessarily be equally true of f, f_1, &c.

Now we know that the symbol (d) is subject to these laws for

$$d^m . d^n(x) = d^{(m+n)}(x)$$

$$\frac{d}{dx}\left(\frac{d}{dy}(z)\right) = \frac{d}{dy}\left(\frac{d}{dx}(z)\right) \text{......}(2)$$

$$d(x) + d(y) = d(x + y),$$

and the same is true for the symbol Δ.

Hence the binomial theorem (to take a particular case) which has been proved for (a) and (b) is equally true for $\left(\frac{d}{dx}\right)$ and $\left(\frac{d}{dy}\right)$: so that we require no farther proof for the proposition, that when u is a function of two independent variables x and y,

$$d^n(u) = \left(\frac{d}{dx}dx + \frac{d}{dy}dy\right)^n u = \frac{d^n u}{dx^n}dx^n + n\frac{d^{n-1}}{dx^{n-1}}\frac{d}{dy}u\,dx^{n-1}dy + \dots$$

But this reasoning will not apply in the case of those functions where the same laws do not hold. For instance, if we take the function log, we have not the condition

$$\log(x) + \log(y) = \log(x + y).$$

But $\qquad \log(x) + \log(y) = \log(xy).$

Consequently the binomial theorem will not hold for this function, though a binomial theorem might possibly be deduced for it, if the expressions did not become so complicated as to be unmanageable.

We have as yet only considered the combinations of operations of one kind, but in the preceding pages we frequently made use of operations of different kinds together, as in the expression $\left(\frac{d}{dx} - a\right)$. Now so long as each of the operations is subject to the same laws, and that they are independent, that is to say, that the one symbol is not supposed to act on the other, the same deductions will follow as when the operations are of the same kind. Hence we assumed that as the expression

$$x^n + Ax^{n-1} + Bx^{n-2} + \text{\&c.} + S$$

can be resolved into the factors

$$(x - a_1)(x - a_2)(x - a_3) \text{ \&c.}$$

The expression

$$\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + \&c. + S$$

can be resolved into the factors

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right) \dots\dots \left(\frac{d}{dx} - a_n\right),$$

which is the foundation of the method we have explained.

But if we have united together such symbols as $\left(\frac{d}{dx} + x\right)$, the same result will not hold. For though (x) is an operation of the same kind as (a), yet it bears a different relation to $\left(\frac{d}{dx}\right)$, as by the nature of this last operation it affects the operation (x), so that

$$x\left(\frac{d}{dx}(z)\right) \text{ is not equal to } \frac{d}{dx}\{x(z)\},$$

or the second law of combination does not hold with regard to these symbols of operation, and, consequently, theorems for other symbols deduced from this law are not true for such symbols as $\left(\frac{d}{dx}\right)$ and (x) together. It is this peculiarity with regard to the combinations of the symbols (x) and $\frac{d}{dx}$ which gives rise to the difficulty in the solution of linear equations with variable coefficients.

Since this article was written, we have learnt that a report by *Cauchy* on *Brisson's* Memoirs, which appears to have been favourable, was rejected by the Academy of Sciences. We know not for what reason.

Γ.

VI.—SOLUTION OF TWO PROBLEMS IN ANALYTICAL GEOMETRY.

In one of the Problem papers for 1836 there is given the following problem: To draw a tangent to a curve of the second order from a point P without it. From P draw any two lines, each cutting the curve in two points. Join the points of intersection two and two, and let the points in which the joining lines (produced if necessary) cross each other be joined by a line which will in general cut the curve in two points A, B. PA, PB are tangents at A and B. This

problem admits of a very elegant solution, which is applicable to many similar questions, and which we shall therefore lay before our readers. Taking the two lines drawn from P as coordinate axes, the equation to the curve is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \ldots\ldots (1).$$

Let the curve cut the axis of x in points M, M′, and the axis of y in points N, N′, and let $PM = a$, $PM' = a'$, $PN = b$, $PN' = b'$.

The equation to the line joining MN is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

the equation to the line joining M′N′ is

$$\frac{x}{a'} + \frac{y}{b'} = 1;$$

and, as at their intersection we may combine their equations in any manner, adding them we get

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2 \ldots\ldots (2).$$

Again, the equation to the line joining MN′ is

$$\frac{x}{a} + \frac{y}{b'} = 1,$$

and the equation to the line joining M′N is

$$\frac{x}{a'} + \frac{y}{b} = 1;$$

at their intersection

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2 \ldots\ldots (3),$$

which is identical with (2), and therefore is the equation to the line joining the points of intersection.

If, now, in equation (1) we make $y = 0$, we get

$$Ax^2 + Dx + F = 0,$$

as the equation for determining a and a'. And, by the theory of equations,

$$a + a' = -\frac{D}{A}, \quad aa' = \frac{F}{A};$$

$$\text{therefore } \frac{1}{a} + \frac{1}{a'} = -\frac{D}{F}.$$

$$\text{Similarly, } \frac{1}{b} + \frac{1}{b'} = -\frac{E}{F};$$

hence equations (2) and (3) become

$$Dx + Ey + 2F = 0 \ldots\ldots\ldots\ldots (4).$$

If, now, we were to change the coordinate axes, retaining the same origin, we should have to make in (1) substitutions of the form

$$x = mx' + ny',$$
$$y = m'x' + n'y'.$$

From the form of these it appears, that those terms which are of the second degree in x and y, would not in their changes affect the other terms; for every term involving x^2, xy, and y^2, would after the change involve only x'^2, $x'y'$, and y'^2. Similarly, the terms which are of the first degree would also change independently. And it is clear that the constant term F would experience no change at all. Now, if we were to make the substitutions in equation (4) the term 2F would, as before, remain the same, and the terms $Dx+Ey$ would suffer the same change as the same terms in the equation to the curve. From this it appears, that if the lines PMM′, PNN′ change their position so that the coordinates are altered, the equation (4), when deduced from the transformed equation (1), will be the same as when the transformation is effected directly on itself; which shows that the line represented by (4) remains fixed in position while the coordinate axes are changed. Now the coordinate axes, cutting the curve each in two points, as they change their position, will ultimately become tangents, and this evidently at the points in which the line, which we have shown to be fixed in position, cuts the curve. Hence follows the method given in the problem.

In one of the problem papers for 1835, the following problem, which may be solved by the same principle, is given: If AB, A′B′ be any two chords in a surface of the second order, the locus of the intersection of AA′, BB′ is a plane.

Take O the point of intersection of AB, A′B′ as origin, OAB as the axis of x, OA′B′ as the axis of y. Put $OA = a$, $OB = a'$, $OA' = b$, $OB' = b'$.

The equation to the surface is

$$Ax^2 + A'y^2 + A''z^2 + Byz + B'xz + B''xy + Cx + C'y + C''z + E = 0 \ldots\ldots\ldots (1);$$

the equation to the line AA′ is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

the equation to the line BB′ is

$$\frac{x}{a'} + \frac{y}{b'} = 1.$$

At their point of intersection we have, by adding the equations,

$$x\left(\frac{1}{a} + \frac{1}{a'}\right) + y\left(\frac{1}{b} + \frac{1}{b'}\right) = 2 \ldots\ldots\ldots\ldots (2).$$

If, now, in equation (1) we make $z = 0$, $y = 0$, we have

$$Ax^2 + Cx + E = 0$$

as the equation for determining a and a'; whence

$$a + a' = -\frac{C}{A}, \quad aa' = \frac{E}{A};$$

$$\text{therefore } \frac{1}{a} + \frac{1}{a'} = -\frac{C}{E}.$$

Similarly we should find

$$\frac{1}{b} + \frac{1}{b'} = -\frac{C'}{E};$$

whence equation (2) becomes

$$Cx + C'y + 2E = 0.$$

Now this equation may be considered as the equation of the line in which the plane whose equation is

$$Cx + C'y + C''z + 2E = 0 \ldots\ldots\ldots (3)$$

cuts the plane of xy.

And, as in the last problem, we may show that the plane represented by equation (3) remains fixed in position; so that the locus of the intersection of AA', BB' is a plane fixed in space, and determined by the equation

$$Cx + C'y + C''z + 2E = 0.$$

R. S.

VII.—PRINCIPAL AXES OF ROTATION.

The moment of inertia of a system about an axis, which passes through the origin of coordinates, and makes angles α, β, γ with the axes of x, y, z, is equal to

$$f \sin^2 \alpha + g \sin^2 \beta + h \sin^2 \gamma$$
$$- 2F \cos\beta . \cos\gamma - 2G \cos\gamma . \cos\alpha - 2H \cos\alpha . \cos\beta,$$
$$\text{where } f = \int x^2 dm, \quad g = \int y^2 dm, \quad h = \int z^2 dm,$$
$$\text{and } F = \int yz dm, \quad G = \int zx dm, \quad H = \int xy dm.$$

This expression will be much simplified if we can assign to the coordinate axes such a position as shall render the corresponding values of F, G, and H equal to zero.

To discover whether this be possible, let us suppose the system referred to three rectangular axes of coordinates x', y', z', which make angles with the axes of x, y, z, whose cosines are a, b, c; a', b', c' and a'', b'', c'' respectively. Between these nine cosines

there are six equations of condition, so that there remain but three to be satisfied in order to determine the position of the new axes. We require then that

$$\int y'z'dm = 0, \quad \int z'x'dm = 0, \quad \int x'y'dm = 0.$$

Now we have

$$x = ax' + a'y' + a''z' \text{.........}(1)$$
$$y = bx' + b'y' + b''z' \text{........}(2)$$
$$z = cx' + c'y' + c''z' \text{.........}(3)$$

and
$$x' = ax + by + cz \text{..... ....}(4).$$

Multiplying (1) by $x'dm$, and integrating, we have

$$\int xx'dm = a\int x'^2dm,$$

the other terms vanishing by the conditions.

Also, multiplying (4) by xdm, and integrating, we have

$$\int xx'dm = a\int x^2dm + b\int xydm + c\int xzdm$$
$$= af + b.\mathrm{H} + c.\mathrm{G}$$
$$= a\int x'^2dm$$

by the last equation.

Putting $\int x'^2dm = \mathrm{X}$, and transposing the terms of the equation, we have

$$a.(f - \mathrm{X}) + b.\mathrm{H} + c.\mathrm{G} = 0 \text{.........}(5).$$

Treating the equations (2) and (4), (3) and (4) in the same manner, putting only y and z respectively in place of x, we obtain

$$a.\mathrm{H} + b.(g - \mathrm{X}) + c.\mathrm{F} = 0 \text{.........}(6)$$
$$a.\mathrm{G} + b.\mathrm{F} + c.(h - \mathrm{X}) = 0 \text{.........}(7),$$

and then eliminating the quantities a, b and c from equations (5), (6) and (7), by the method of cross multiplication, we obtain the equation

$$(f - \mathrm{X})(g - \mathrm{X})(h - \mathrm{X})$$
$$- \mathrm{F}^2.(f - \mathrm{X}) - \mathrm{G}^2(g - \mathrm{X}) - \mathrm{H}^2(h - \mathrm{X}) - 2\mathrm{FGH} = 0,$$

which is that arrived at p. 267, Whewell's *Dynamics*. We should have obtained the same equation for $\mathrm{Y} = \int y'^2dm$, or for $\mathrm{Z} = \int z'^2dm$, and therefore, as it may be shown that the three roots of the cubic are real, they are the values of X, Y, and Z.

P.

VIII.—ANALYTICAL GEOMETRY OF THREE DIMENSIONS.

No. I.

THE series of articles which we intend to give on this subject are chiefly designed to exhibit the advantages of mathematical symmetry. The French writers have commonly been more attentive to this than our own, and it may be seen frequently exemplified in Leroy's *Analyse appliquée a la Géometrie des trois dimensions.* But neither he, nor any other person that we know of, has made any use of the following symmetrical form of the equations to the straight line, though he mentions it, page 17, 2nd edit.

1. Let x', y', z' be the coordinates of any fixed point through which the line passes; and λ, μ, ν, the angles which it makes with the axes of coordinates, supposed rectangular, then

$$\frac{x-x'}{\cos\lambda}=\frac{y-y'}{\cos\mu}=\frac{z-z'}{\cos\nu}\text{ ....(1)}$$

are the equations to the line.

If the angles between the coordinates be any whatever, and if l, m, n, be the ratios of the projections of any portion of the line on the axes of coordinates, to that portion, the projections being made by planes parallel to the coordinate planes, then the equations to the line are

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}.$$

The demonstration of this is very simple.

Let r be the length of the portion of the line between the points (x', y', z'), (x, y, z), then the projections of r are

$$lr,\ mr,\ nr.$$

But these projections are also

$$x-x',\ y-y',\ z-z'.$$

Equating these values, we obtain

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r\text{ ....(2).}$$

It will frequently be convenient to introduce the quantity r in investigations.

2. If the angles between yz, zx, xy be denoted by yz, zx, xy respectively,

$$r^2=x^2+y^2+z^2+2yz\cos yz+2zx\cos zx+2xy\cos xy\text{ ....(3),}$$

therefore

$$l^2+m^2+n^2+2mn\cos yz+2nl\cos zx+2lm\cos xy=1\text{ ....(4)}$$

is the relation connecting l, m, n.

3. It is evident that if L, M, N be any quantities respectively proportional to l, m, n, the equations to the line may be written,

$$\frac{x-x'}{L} = \frac{y-y'}{M} = \frac{z-z'}{N}.$$

4. We shall not stop to prove the expression for the cosine of the angle contained between two lines, in terms of the angles which they make with a system of rectangular coordinates, but proceed to the more general proposition of finding the angle between two given lines, the angles between the coordinates being any whatever.

Let the equations to two straight lines parallel to the given lines and passing through the origin, be

$$\left.\begin{aligned} \frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r \\ \frac{x'}{l'} = \frac{y'}{m'} = \frac{z'}{n'} = r' \end{aligned}\right\} \ldots\ldots\ \ldots\ldots (5),$$

and let θ be the angle between them.

Then the square of the distance between the extremities of the lines r, r', is

$$r^2 - 2rr' \cos\theta + r'^2 \ldots\ldots\ldots\ldots\ldots (6);$$

but it is also the same as the square of the distance between the points (x, y, z), (x', y', z'), which by the expression (3), § 2, is

$$\begin{aligned} &(x-x')^2 + (y-y')^2 + (z-z')^2 \\ &+ 2(y-y')(z-z')\cos yz + 2(z-z')(x-x')\cos zx \\ &+ 2(x-x')(y-y')\cos xy \\ &= r^2 + r'^2 - 2\{xx' + yy' + zz' + (yz' + y'z)\cos yz \\ &+ (zx' + z'x)\cos zx + (xy' + x'y)\cos xy\} \ldots\ldots (7). \end{aligned}$$

Equating the expressions (6) and (7), substituting for x, y, z, x', y', z', in terms of r and r' from equations (5), and reducing, we get

$$\begin{aligned} \cos\theta = ll' + mm' + nn' + (mn' + m'n)\cos yz + (nl' + n'l)\cos zx \\ + (lm' + l'm)\cos xy \ldots\ldots\ldots\ldots (8), \end{aligned}$$

which is one expression for $\cos\theta$.

If the two lines coincide, so that $\theta = 0$, $l' = l$, $m' = m$, and $n' = n$, equation (8) becomes the same as the equation (4) expressing the relation between l, m, n.

If the axes be rectangular, $\cos yz$, $\cos zx$, $\cos xy$, are each $= 0$, and (8) becomes

$$\cos\theta = ll' + mm' + nn',$$

l, m, n, l', m', n' becoming in this case the cosines of the angles which the two lines make with the axes.

5. Let the angles which the first line makes with the axes, in the general case, be λ, μ, ν, we may deduce from (8) the values of λ, μ, ν, in terms of l, m, n.

For suppose the second line to coincide with the axis of x, then θ becomes λ; and by considering the signification of these quantities, it will be seen that l' becomes 1, m' and n' each become 0. Hence (8) reduces to

$$\left.\begin{aligned} \cos\lambda &= l + n\cos zx + m\cos xy. \\ \text{In like manner,}\quad \cos\mu &= m + n\cos yz + l\cos xy, \\ \text{and}\quad \cos\nu &= n + m\cos yz + l\cos zx. \end{aligned}\right\} \dots\dots\dots(9).$$

6. We may now obtain a shorter expression for $\cos\theta$. Multiply equations (9) by l', m', n' respectively, and add the products; then the second member of the resulting equation will be identical with the second member of (8), therefore

$$\cos\theta = l'\cos\lambda + m'\cos\mu + n'\cos\nu \dots\dots\dots(10).$$

This equation is remarkable for being of the same form as when the axes are rectangular. By supposing the two lines to coincide, we obtain from it

$$l\cos\lambda + m\cos\mu + n\cos\nu = 1 \dots\dots\dots\dots(11),$$

a relation between these quantities independent of the angles between the axes.

7. The values of l, m, n, in terms of λ, μ, ν, may be easily found from equations (9,) by cross multiplication.

We may also substitute the values of l', m', n', in terms of λ', μ', ν', in (10), and thus obtain an expression for $\cos\theta$ in terms of the angles which the lines make with the axes.

8. From (10) we obtain for the condition, that a line which makes angles λ, μ, ν with the axes, may be at right angles to a line whose equations are

$$\frac{x - x'}{l'} = \frac{y - y'}{m'} = \frac{z - z'}{n'},$$

$$l'\cos\lambda + m'\cos\mu + n'\cos\nu = 0 \dots\dots\dots(12).$$

Another form of the condition may be obtained from (8).

9. To obtain the equation to the plane, we shall consider it as generated by the motion of a straight line which always meets a fixed straight line, and remains parallel to a given position.

Let the equations to the generating line be

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} = r,$$

and those to the fixed line,

$$\frac{x' - a}{l'} = \frac{y' - b}{m'} = \frac{z' - c}{n'} = r'.$$

Hence,

$$x - x' = lr, \quad y - y' = mr, \quad z - z' = nr;$$
$$x' - a = l'r', \quad y' - b = m'r', \quad z' - c = n'r'.$$

Adding these equations, so as to eliminate x', y', z',

$$\left.\begin{aligned} x - a &= lr + l'r', \\ y - b &= mr + m'r' \\ z - c &= nr + n'r' \end{aligned}\right\} \dots\dots\dots (13).$$

Let α, β, γ be the angles which a line perpendicular to both the former, makes with the axes, so that, according to (12),

$$\left.\begin{aligned} l \cos \alpha + m \cos \beta + n \cos \gamma &= 0, \\ l' \cos \alpha + m' \cos \beta + n' \cos \gamma &= 0, \end{aligned}\right\} \dots\dots\dots (14);$$

then, multiplying equations (13) by $\cos \alpha$, $\cos \beta$, $\cos \gamma$, respectively, and adding, r and r' disappear, and

$$(x - a) \cos \alpha + (y - b) \cos \beta + (z - c) \cos \gamma = 0 \dots\dots\dots (15),$$

which is the equation to the plane. This equation proves Euclid, book XI. prop. 4, for if ρ be the distance of the points (x, y, z) and (a, b, c), $x - a$, $y - b$, $z - c$, are the projections of ρ on the axes, and the equation

$$\frac{x - a}{\rho} \cos \alpha + \frac{y - b}{\rho} \cos \beta + \frac{z - c}{\rho} \cos \gamma = 0$$

shews that the line which makes angles α, β, γ with the axes is perpendicular to ρ, that is, to any line which meets it in that plane, and therefore it is perpendicular to the plane.

10. It is easy to derive from (15) other useful forms of the equation to the plane. We may suppose a, b, c, to be the coordinates of the point where the perpendicular from the origin on the plane meets it; then if the length of the perpendicular be δ, we have by (11), since a, b, c are the projections of δ upon the axes,

$$\frac{a}{\delta} \cos \alpha + \frac{b}{\delta} \cos \beta + \frac{c}{\delta} \cos \gamma = 1,$$

whence

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \delta \dots\dots\dots (16).$$

11. The sine of the angle between the line whose equations are

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n},$$

and the plane whose equation is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \delta,$$

is the same as the cosine of the angle between the line and the perpendicular to the plane, which, by (10), is

$$l \cos \alpha + m \cos \beta + n \cos \gamma.$$

If the line be parallel to the plane, this must be equal to 0. If the line be situated in the plane, we must have, in addition,

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma = \delta.$$

When the coordinates are rectangular, the conditions that the line may be perpendicular to the plane, are

$$l = \cos \alpha, \quad m = \cos \beta, \quad n = \cos \gamma.$$

12. To find the perpendicular distance from a given point (x', y', z') to the plane represented by

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \delta.$$

Let a parallel plane be drawn through the point (x', y', z'), then its equation will be

$$(x - x') \cos \alpha + (y - y') \cos \beta + (z - z') \cos \gamma = 0.$$

The required perpendicular will evidently be equal to the difference between the perpendiculars from the origin on these two planes

$$= (x' \cos \alpha + y' \cos \beta + z' \cos \gamma) \sim \delta.$$

13. To find the shortest distance between two lines, whose equations are

$$\left.\begin{aligned} \frac{x-a}{l} &= \frac{y-b}{m} = \frac{z-c}{n} = r \\ \frac{x'-a'}{l'} &= \frac{y'-b'}{m'} = \frac{z'-c'}{n'} = r' \end{aligned}\right\} \dots\dots\dots (17),$$

referred to rectangular coordinates.

If D be the distance of two points in the two lines,

$$D^2 = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

Now x, y, z are each functions of r, and x', y', z' of r'; but r and r' are independent, therefore, when D is a minimum,

$$(x - x')\, dx + (y - y')\, dy + (z - z')\, dz = 0,$$
$$(x - x')\, dx' + (y - y')\, dy' + (z - z')\, dz' = 0.$$

But

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n} = dr,$$

$$\frac{dx'}{l'} = \frac{dy'}{m'} = \frac{dz'}{n'} = dr'.$$

Substituting in the preceding equations,

$$\left.\begin{aligned} l(x - x') + m(y - y') + n(z - z') &= 0 \\ l'(x - x') + m'(y - y') + n'(z - z') &= 0 \end{aligned}\right\} \dots\dots (18).$$

From these equations we see that the coordinates of the extremities of the shortest distance satisfy the equations to two planes, respectively perpendicular to the two given lines. The line of the shortest distance is therefore the intersection of these two planes, and therefore perpendicular to both the given lines.

The six equations (17) and (18) determine the values of the six coordinates x, y, z, x', y', z'. The best way to solve them would be to substitute in (18) the values of the coordinates in terms of r and r' obtained from (17): thus we should have two simple equations for determining r and r', the values of which being found, those of the coordinates would be known.

The value of the least distance may easily be determined by supposing two planes, parallel to one another, to pass through the given lines. These planes will therefore be perpendicular to the least distance, and their equations will be

$$(x-a)\cos\alpha + (y-b)\cos\beta + (z-c)\cos\gamma = 0,$$
$$(x-a')\cos\alpha + (y-b')\cos\beta + (z-c')\cos\gamma = 0,$$

$\cos\alpha$, $\cos\beta$, and $\cos\gamma$ being determined by the equations

$$l\cos\alpha + m\cos\beta + n\cos\gamma = 0,$$
$$l'\cos\alpha + m'\cos\beta + n'\cos\gamma = 0,$$
$$(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1.$$

The least distance of the lines will be the perpendicular distance of these parallel planes, or the difference of the perpendiculars upon them from the origin, which is

$$\pm\{(a-a')\cos\alpha + (b-b')\cos\beta + (c-c')\cos\gamma\}.$$

ff.

IX.—NOTE ON THE THEORY OF THE SPINNING-TOP.

THE manner in which friction causes a spinning-top to raise itself into a vertical position, has never, as far as I know, been distinctly shown. Euler gives the following explanation, which will be found in a note to Whewell's *Dynamics*, p. 324. "The friction will perpetually retard the motion of the apex of the instrument, and at last reduce it to rest. If this happen before the top fall, it must then be spinning in such a position that the point can remain stationary; but this cannot be if it be inclined. Hence it must have a tendency to erect itself into a vertical position." This reasoning is not only of the most vague and inconclusive kind, but is remarkable as being directly the reverse of the truth. For when the friction acts so as to retard the apex, it tends to make the top fall; and it is only when the friction accelerates the apex, that it causes the top to raise itself into a vertical position.

It is well known, that if a body have velocities communicated to it about different axes, which are represented by distances taken along these axes, proportional to the velocities, all the rotations being considered as "right-handed" or "left-handed" rotations, being represented by taking the distances in the negative direction, then the axis and magnitude of the resultant rotation will be represented by the resultant of these lines combined as lines representing forces.

Suppose, now, a top, whose apex is a mathematical point, to be spinning on a smooth horizontal plane in an inclined position, and, for distinctness of conception, let this be the plane of xy, and let the axis be in the plane yz.

The action of gravity will tend to make it fall, by giving it a motion of rotation about an axis parallel to that of x. The angular velocity this would communicate in an instant of time may be represented by a line in the direction of $-x$. The combination of these two will cause the instantaneous axis to move a little towards the axis of $-x$, and the axis of figure to follow *nearly* in the same direction, and the effect will be a precessional motion of the top in the same direction as the rotation, combined with a nutation which we do not here consider. Consequently, the centre of gravity remaining in the same vertical line, the apex will describe a circular path "with the Sun."

If the plane be rough, the apex will endeavour to describe the same path, but will be constantly retarded by friction, which at the instant we consider will be a force pulling the apex in the direction of $-x$, which will tend to produce a rotation about an axis perpendicular to a plane passing through the direction and the centre of gravity, or lying between the axes of y and $-z$. The combination of this with the original rotation will thus cause the axis of the top to approach the axis of y, or to fall.

Hitherto we have supposed the apex of the top to be a mathematical point, which can never be exactly the case; the real apex will be a surface. If we consider the top as terminated by a portion of a sphere, it will be seen that the top spins on no particular point, but that each point in the circumference of a small circular curve is successively in contact with the plane, the diameter of the curve increasing with the inclination of the top.

Conceive, now, the top spinning in its original position, and with no precessional motion, the apex will endeavour to roll along the axis of x; if it is prevented from rolling, there will be a rubbing friction tending to pull the apex in that direction.

Suppose, now, the top to spin freely—the precessional motion makes the apex move towards x: if this velocity is equal to the rolling velocity, there will be no friction called into action, and the top will spin as if on a smooth plane. If the precessional motion be greater than the rolling motion, there will be a retarding friction which will cause the top to fall. *But if the rolling motion is quicker than the precessional, there will be an accelerating friction which will tend to raise the top.*

It is easy to show, by means of a teetotum, that this theory agrees with experiment. If the apex be cut to a point, no velocity we can communicate will make it spin upright; if the apex is rounded, as it usually is, the instrument will rise at first, and then fall gradually, the increased diameter of the circle in which it touches the plane compensating for the diminishing velocity of rotation and the increasing velocity of precession: or if the end be a cylinder cut

perpendicularly to its axis, in which case this compensation does not take place, it will be found that the teetotum spins upright for some time, and then falls very suddenly as soon as the direction of friction changes.

H. T.

X.—ON THE SOLUTION OF CERTAIN TRIGONOMETRICAL EQUATIONS.

THERE are several trigonometrical equations whose roots can be readily obtained by taking into consideration their connection with the general binomial equation whose last term is unity. If for example we have the equation

$$\cos\theta + \cos 2\theta + \cos 3\theta + \dots + \cos(n-1)\theta = 0,$$

we can find its roots by means of the equation

$$x^{2n} - 1 = 0.$$

We know that the roots of this last equation are of the form

$$\pm 1,\ \cos\phi \pm \sqrt{-1}\sin\phi,\ \cos 2\phi \pm \sqrt{-1}\sin 2\phi \dots$$
$$\cos(n-1)\phi \pm \sqrt{-1}\sin(n-1)\phi.$$

Now as the equation wants the second term, the sum of its roots must be 0; and as the possible and impossible parts do not affect each other, they must be separately equal to 0; also, as the roots $+1$ and -1 destroy each other, there remains

$$\cos\phi + \cos 2\phi + \cos 3\phi + \dots \cos(n-1)\phi = 0.$$

Comparing this with the original equation, we see that the former will be satisfied by making $\theta = \phi$. But to determine ϕ, we have the equation

$$(\cos\phi + \sqrt{-1}\sin\phi)^{2n} - 1 = 0,$$

$$\text{or } \cos 2n\phi + \sqrt{-1}\sin 2n\phi = 1.$$

Whence, as the possible and impossible parts are independent,

$$\cos 2n\phi = 1, \quad \sin 2n\phi = 0;$$

$$\text{which give } 2n\phi = 2m\pi,$$

$$\text{or } \phi = \frac{m\pi}{n},$$

where m has any integer value from 0 to n, making in all $n+1$ values of ϕ. But two of these cannot be taken as values of θ, as we excluded the roots $+1$ and -1, which correspond to the values 0 and n of m. So that θ will be found from the equation

$$\theta = \frac{m\pi}{n},$$

where m has any value from 1 to $n-1$, making in all $n-1$ values

of θ which answer the given equation. In exactly the same way we might shew how to solve the equation

$$\cos\theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2n-1)\theta = 0.$$

For its roots would be deduced from the equation

$$(\cos\theta + \sqrt{-1}\sin\theta)^{2n} + 1 = 0,$$

$$\text{or} \quad \cos 2n\theta + \sqrt{-1}\sin 2n\theta = -1;$$

which gives $\qquad \cos 2n\theta = -1, \quad \sin 2n\theta = 0.$

Therefore $\qquad 2n\theta = (2m+1)\pi,$

$$\text{and } \theta = \frac{(2m+1)}{2n}\pi;$$

where m has any value from 0 to $n-1$, giving on the whole n values of θ which satisfy the equation.

Similarly, the equations

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta = 0,$$

$$\text{and } \cos\theta + \cos 3\theta + \dots + \cos(2n-1)\theta = 1,$$

may be solved by means of the equations

$$x^{2n+1} - 1 = 0,$$

$$\text{and } x^{2n+1} + 1 = 0.$$

For the first we shall have

$$\theta = \frac{2m\pi}{2n+1},$$

where m has n values from 1 to n.

For the second we shall have

$$\theta = \frac{2m+1}{2n+1}\pi,$$

where m has n values from 0 to $n-1$.

The same method may be extended to other equations. For the roots of $x^{2n} - 1 = 0$ being of the form

$$\cos\phi + \sqrt{-1}\sin\phi.$$

If we multiply each by $\cos a + \sqrt{-1}\sin a$, it becomes

$$\cos(a+\phi) + \sqrt{-1}\sin(a+\phi).$$

But the sum of the roots will still remain equal to 0 when multiplied by $\cos a + \sqrt{-1}\sin a$, and taking away the terms which destroy each other, there will remain

$$\cos(a+\phi) + \cos(a+2\phi) + \cos(a+3\phi) + \dots$$
$$+ \cos\{a + (n-1)\phi\} = 0,$$

consequently the equation

$$\cos(a+\theta) + \cos(a+2\theta) + \dots + \cos\{a+(n-1)\theta\} = 0,$$

will be satisfied by the same values of θ as the first of the given equations; and similarly we might proceed with the others.

F.

XI.—MATHEMATICAL NOTES.

UNDER this head we propose to insert new demonstrations of known theorems, solutions of interesting problems, and in general short notices of methods of mathematical investigation which are likely to be useful to all classes of students in this university.

1. *Elimination by means of Cross Multiplication.* This is a convenient mnemonic rule for elimination, applications of which continually occur in various branches of analysis; and references to which have been frequently made in different articles of this number.

If we have three symmetrical equations,

$$Ax + By + Cz = D \ldots\ldots\ldots\ldots\ldots(1).$$
$$A_1x + B_1y + C_1z = D_1 \ldots\ldots\ldots\ldots\ldots(2),$$
$$A_2x + B_2y + C_2z = D_2 \ldots\ldots\ldots\ldots\ldots(3),$$

between which we wish to eliminate y and z; we can perform the operation at once by the following rule:

Multiply (1) by $B_1C_2 - B_2C_1$,
(2) by $B_2C - BC_2$,
(3) by $BC_1 - B_1C$,

and add. It will be found on trial, that the terms involving y and z disappear, and we obtain

$$\{A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C)\}\,x$$
$$= D(B_1C_2 - B_2C_1) + D_1(B_2C - BC_2) + D_2(BC_1 - B_1C),$$

from which x is known. And in a similar manner we might determine y and z by eliminating x and z, and x and y successively.

If $D = 0$, $D_1 = 0$, $D_2 = 0$, the second side of the equation disappears, and as x divides out, we have, as the result of the elimination of x, y, z, the equation

$$A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C) = 0.$$

As examples of cases in which this method is of great advantage, we may notice the investigation of the cubic equation of condition for the existence of three principal axes of rotation, and of three principal diametral planes in surfaces of the second order, and the demonstration of the properties of conjugate diameters in these surfaces. It is also of great use, in finding the equation to the osculating plane, and the radius of absolute curvature, if all the expressions be put in a symmetrical form. The form of the multipliers may be easily deduced by Lagrange's method of indeterminate multipliers, and their symmetry greatly facilitates the practical application of the method.

σ.

2. *To find $\frac{d\theta}{de}$ and $\frac{dr}{de}$ in the Planetary Theory.* The following method, by the introduction of a subsidiary quantity, simplifies greatly the analytical operations, and more particularly avoids a very troublesome integration. (*Airy*, p. 94, *Pratt*, p. 331.)

Taking the equations of elliptic motion,

$$nt = u - e \sin u \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\tan \frac{u}{2} = \frac{\sqrt{1-e}}{\sqrt{1+e}} \tan\left(\frac{\theta - \omega}{2}\right) \ldots\ldots\ldots(2).$$

Differentiate (1) with regard to e, considering t as constant, and take the logarithmic differential of (2) with regard to the same variable. Then we have

$$0 = \frac{du}{de}(1 - e\cos u) - \sin u \ldots\ldots\ldots\ldots(3),$$

$$\frac{1}{\sin u}\,\frac{du}{de} = -\frac{1}{1-e^2} + \frac{1}{\sin(\theta-\omega)}\,\frac{d\theta}{de} \ldots\ldots(4).$$

Eliminating $\frac{1}{\sin u}\,\frac{du}{de}$, and observing that

$$1 - e\cos u = \frac{1-e^2}{1 + e\cos(\theta - \omega)},$$

we get $$\frac{1}{\sin(\theta-\omega)}\,\frac{d\theta}{de} = \frac{2 + e\cos(\theta-\omega)}{1-e^2};$$

whence $$\frac{d\theta}{de} = \frac{\sin(\theta-\omega)\,\{2 + e\cos(\theta-\omega)\}}{1-e^2}.$$

Again, we have $\quad r = a(1 - e\cos u).$

Therefore $$\frac{dr}{de} = -a\cos u + ae\sin u\,\frac{du}{de}.$$

Eliminating $\frac{du}{de}$ by means of (3),

$$\frac{dr}{de} = -a\cos u + \frac{ae\sin^2 u}{1 - e\cos u} = -\frac{a\cos u + ae}{1 - e\cos u}.$$

Therefore $$\frac{dr}{de} = -\frac{a(\cos u - e)}{1 - e\cos u} = -a\cos(\theta - \omega).$$

π.

3. *Evolute to the Ellipse.* The equation to the evolute of the ellipse may be found very readily by considering it as the locus of the ultimate intersection of consecutive normals.

Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;.\;.\;.\;.\;.\; (1)$$

be the equation to the ellipse. Then the equation to a normal passing through a point x, y, will be

$$\frac{b^2(\beta-y)}{y} - \frac{a^2(\alpha-x)}{x} = 0,$$

$$\text{or } \frac{b^2\beta}{y} - \frac{a^2\alpha}{x} = a^2 - b^2 \quad . \; . \; . \; . \; . \; (2),$$

where α and β are the coordinates of the normal itself. To find the locus of the ultimate intersection of the normals, we must differentiate considering α and β as constant, x and y as variable. We then have from equations (1) and (2)

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} = 0 \quad . \; . \; . \; . \; . \; (3)$$

$$\frac{b^2\beta dy}{y^2} - \frac{a^2\alpha dx}{x^2} = 0 \quad . \; . \; . \; . \; . \; (4)$$

λ (3) + (4) gives, on equating to 0 the coefficients of each differential,

$$\frac{\lambda x}{a^2} = -\frac{a^2\alpha}{x^2} \cdot \frac{\lambda y}{b^2} = \frac{b^2\beta}{y^2}.$$

Multiply the first of these by x, and the second by y, and add.

Then $\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \lambda = \frac{b^2\beta}{y} - \frac{a^2\alpha}{x} = a^2 - b^2.$

Substituting

$$\frac{a^2-b^2}{a^2} x = -\frac{a^2\alpha}{x^2} \cdot \frac{a^2-b^2}{b^2} y = \frac{ba^2\beta}{y^2}.$$

Therefore $\frac{x^3}{a^3} = -\frac{a\alpha}{a^2-b^2} \cdot \frac{y^3}{b^3} = \frac{b\beta}{a^2-b^2},$

and these values of $\frac{x}{a}$ and $\frac{y}{b}$ being substituted in the equation to the ellipse, give

$$a^{\frac{2}{3}}\alpha^{\frac{2}{3}} + b^{\frac{2}{3}}\beta^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

γ.

ERRATUM.

Page 4, line 10, *dele* the words "the force of restitution may be taken as proportional to it, so that."